The Borel reducibility Main Gap

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The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1965:** Morley's categoricity theorem.

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T,\lambda) \leq I(T,\kappa).$$

1990: Shelah proved Morley's conjecture.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha) = 2^\alpha$*; or* $\forall \alpha > 0$, $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1991:** Väänänen Cantor-Bendixson theorem.
- **2014:** Friedman-Hyttinen-Kulikov developed GDST and a systematic comparison between the Main Gap dividing lines and the complexity given by Borel reducibility.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Let $\omega \leq \mu \leq \kappa$ be a cardinal. Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π_μ between $\mu^{<\omega}$ and μ.

Definition

For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_{\eta\restriction\mu}$ with domain μ as follows: For every tuple $(a_1, a_2, \ldots, a_n)$ in μ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\mu}} \Leftrightarrow \eta(\pi_\mu(m, a_1, a_2, \ldots, a_n)) > 0.$$

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $f, g \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

- $\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}$
- $\mathcal{A}_{\eta\restriction\mu} \not\models T, \mathcal{A}_{\xi\restriction\mu} \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

With Borel functions, we can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \text{ iff } \cong_T \hookrightarrow_B \cong_{T'}$$

Question

Question: What can we say about the Borel-reducibility between different dividing lines?

Conjecture: If T is classifiable and T' is not classifiable, then

$$\cong_T \hookrightarrow_B \cong_{T'}.$$

Borel-reducibility Main Gap

Theorem (M.)

Let $\mathfrak{c} = 2^\omega$. *Suppose* $\kappa = \lambda^+ = 2^\lambda$ *and* $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. *If* T *is a classifiable theory, and* T' *is a non-classifiable theory, then*

$$\cong_T \hookrightarrow_C \cong_{T'} \text{ and } \cong_{T'} \not\hookrightarrow_B \cong_T .$$

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_L =^2_\gamma$.

$=^2_\gamma \hookrightarrow_C \cong_T$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\theta$	$2^\mathfrak{c} \leq \lambda = \lambda^\theta$	$2^\mathfrak{c} \leq \lambda = \lambda^{<\lambda}$
Stable Unsuperstable	$\gamma = \omega$	$\gamma = \omega$	$\gamma = \omega$
Unstable	$\omega \leq \gamma \leq \theta$	$\omega \leq \gamma \leq \theta$	$\omega \leq \gamma \leq \lambda$
Superstable with OTOP	$\omega \leq \gamma \leq \theta$	$\omega \leq \gamma \leq \theta$	$\omega \leq \gamma \leq \lambda$
Superstable with DOP	?	$\omega_1 \leq \gamma \leq \theta$	$\omega_1 \leq \gamma \leq \lambda$

Main Gap Dichotomy

Theorem (M.)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta_1^1(\kappa)$;
- $\cong_T$ *is* $\Sigma_1^1(\kappa)$*-complete.*

Classifiable theories

- shallow,

$$I(T, \aleph_\alpha) < \beth_{\omega_1}(|\ \alpha\ |);$$

- non-shallow,

$$I(T, \alpha) = 2^\alpha.$$

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong_T \hookrightarrow_B \cong_{T'}$$

General reduction

Fact (Mangraviti-Motto Ros)

Let E_1 be a Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.

In between

Lemma (M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$. *Let* $\kappa = \aleph_\gamma$ *be such that* $\beth_{\omega_1}(|\gamma|) \leq \kappa$ *and* $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. *Suppose* T_1 *is a classifiable shallow theory,* T_2 *a classifiable non-shallow theory, and* T_3 *non-classifiable theory. Then*

$$\cong_{T_1} \hookrightarrow_B \cong^\lambda_{T_3} \hookrightarrow_C \cong_{T_2} \hookrightarrow_C \cong_{T_3} .$$

Thank you

Article at: https://arxiv.org/abs/2308.07510

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