Borel sets and the generalized Baire spaces

Miguel Moreno
University of Helsinki
(joint work with Tapani Hyttinen and Jouko Väänänen)

Finnish Mathematical Days 2026

9 January, 2026

European Research Council
Established by the European Commission
ACADEMY OF FINLAND

The bounded topology

Let κ be an uncountable cardinal.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

κ-Borel sets

The κ-Borel space of κ^κ is the smallest set, which contains the open sets, and is closed under unions and intersections, both of length κ, and complements.

A κ-Borel set, is any element of the κ-Borel space.

A set X is a κ-Σ_1^1 if it is a projection of a κ-Borel set $C \subseteq (\kappa^\kappa)^2$.

A set X is a κ-Π_1^1 if it is the complement of a κ-Σ_1^1 set.

A set X is a κ-Δ_1^1 if it is a κ-Σ_1^1 and κ-Π_1^1 set.

Classifiable theories

Theorem (Friedman - Hyttinen - Kulikov)

If $\kappa > 2^\omega$ is a succesor cardinal, then T is classifiable and shallow if and only if $\cong_T$ is κ-Borel.

Theorem (Friedman - Hyttinen - Kulikov)

If T is classifiable not shallow, then $\cong_T$ is κ-Δ_1^1 and not κ-Borel.

Some non-Classifiable theories

Theorem (Friedman - Hyttinen - Kulikov)

- *If T is unstable, then* $\cong_T$ *is* κ-Σ_1^1 *and not* κ-Δ_1^1.
- *If T is superstable with OTOP, then* $\cong_T$ *is* κ-Σ_1^1 *and not* κ-Δ_1^1.
- *If T is superstable with DOP and* $\kappa > \omega_1$, *then* $\cong_T$ *is* κ-Σ_1^1 *and not* κ-Δ_1^1.

Borel-reducibility Main Gap

Theorem (M.)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then

$$\cong_T \hookrightarrow_C \cong_{T'} \text{ and } \cong_{T'} \not\hookrightarrow_B \cong_T .$$

A dead end

Lemma (Hyttinen - M - Väänänen.)

Suppose $2^\lambda = 2^\kappa$, $\kappa > \lambda$. Then for all $A \subseteq \kappa^\kappa$ there is a closed $C \subseteq \kappa^\kappa \times 2^\kappa$ such that A is the first projection of C i.e.

$$A = \{f \in \kappa^\kappa : \exists g \in 2^\kappa((f,g) \in C)\}.$$

Ideal topologies

Let $\mathcal{I}$ be a κ-complete proper ideal on κ that extends the ideal of bounded sets.

For any set $I \subseteq \kappa$, let $Fn_I = \{f \mid f : D \to \kappa, \text{ where } D \in I\}$.

For $\eta \in Fn_I$, we define

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}.$$

More issues

Lemma (M - Pitton.)

If $\mathcal{I}$ contains an unbounded subset of κ. Then for all $A \subseteq \kappa^\kappa$ there is a closed $C \subseteq \kappa^\kappa \times \kappa^\kappa$ such that A is the first projection of C.

Definition

Basic κ-open sets are the sets of the form

$$N_\eta = \{\zeta \in \kappa^\kappa \mid \eta \subseteq \zeta\}$$

where $\eta : X \to \kappa$ and $X \subsetneq \kappa$ has size less than κ; and the empty set. A set is κ-open if it is a κ-union of basic *κ-open* sets.

The class of *κ-Borel* sets is the smallest class that contains all the basic κ-open sets and is closed under complements and κ-unions and κ-intersections.

Independency

Lemma (M - Pitton.)

The class of κ-Borel sets is the same over all the ideal topologies.

κ-analytic

We say that a set is κ-Σ_1^1 if it is the projection of a κ-Borel set.

Notice that there are only 2^κ many κ-Σ_1^1 sets, but there are 2^{2^κ} subsets of κ^κ. Most subsets of κ^κ are not κ-Σ_1^1.

Lemma (Hyttinen - M - Väänänen.)

Every κ-Borel set is a κ-Σ_1^1 set.

κ-Borel hierarchy

- κ-Σ_0 is the set of basic κ-open sets of the form N_η, $|\eta| = 1$.
- If α is even, then κ-$\Sigma_{\alpha+1}$ is the set of all κ-unions of κ-Σ_α-sets.
- If α is odd, then κ-$\Sigma_{\alpha+1}$ is the set of all κ-intersections of κ-Σ_α-sets.
- If α is limit, then κ-$\Sigma_\alpha = \cup_{\beta<\alpha}\kappa$-$\Sigma_\beta$.

Good news

Theorem (Hyttinen - M - Väänänen.)

The sets κ-Σ_α form a proper hierarchy.

Theorem (Hyttinen - M - Väänänen.)

For all $\alpha < \kappa^+$, there is a universal κ-Σ_α set.

Theorem (Hyttinen - M - Väänänen.)

There is a universal κ-Borel set.

The classifiable case

Theorem (Hyttinen - M - Väänänen.)

Let $\mathcal{T}$ be a countable ω-stable NDOP shallow. Suppose κ is a regular cardinal such that $\kappa^\omega = \kappa$. If the number of isomorphism classes is at most κ, then $\cong_\mathcal{T}$ is κ-Borel.

The non-classifiable case

Theorem (Hyttinen - M - Väänänen.)

Suppose $\mathcal{T}$ is a countable non-classifiable theory (i.e. $\mathcal{T}$ is not a superstable stable theory without DOP and without OTOP) and $\kappa > \omega$ is regular, then the isomorphism relation of $\mathcal{T}$, $\cong_{\mathcal{T}}$, is not κ-Borel on the Generalized Baire Space κ^κ.

Thank you

Article at: https://arxiv.org/abs/2507.15427