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The Stationary Tower Forcing

Master Thesis

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Abstract

The purpose of this work was to understand the stationary tower forcing ($\mathbb{P}_{<\kappa}$), introduced by W. Hugh Woodin, and the possible suborders (mainly the towers $\mathbb{Q}^S_{<\kappa}$) of $\mathbb{P}_{<\kappa}$, that satisfy the basic properties of $\mathbb{P}_{<\kappa}$, like projection, lifting and normality. Another aim was to study how large cardinal properties of κ influence the forcing with these orders, and finally to study the applications of these forcings.

We defined the operators projection (S_X) and lifting (S^Y) for stationary sets S, and the stationary tower as the partial order with conditions in V_κ that are stationary in the sense of generalized stationarity. The partial order is given by the lifting operator. The tower $\mathbb{Q}^S_{<\kappa}$ is the suborder of $\mathbb{P}_{<\kappa}$ where the conditions are only the substes of S. In most applications $S = \mathcal{P}_\lambda(V_\delta)$ for some regular $\lambda < \delta$.

The results of the study show that if $\mathbb{Q}^S_{<\kappa}$ is closed under projection and lifting, then $\mathbb{Q}^S_{<\kappa}$ has, with slight differences almost all the properties of $\mathbb{P}_{<\kappa}$. we present several important applications of these forcings. The first kind is absolutness results for set forcings from a proper class of woodin cardinals. Another application characterices forcing axioms by embedding into a stationary tower.

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Chapter 1

Introduction

> " *You could think I'm wrong, but that's no reason to stop thinking.*
>
> - Gregory House "

The stationary tower forcing is a method invented by W. Hugh Woodin motivated by the work of Foreman, Magidor and Shelah [FMS88]. It is used most of the times to prove absoluteness results, other applications include characterization of the forcing axioms and derived models.

One of the first results proved by Woodin using the stationary tower was the Σ_1^2-absolutness theorem in 1985.
In 1970 Solovay proved the classical result: *If κ is an inaccessible cardinal and G is a V-generic contained in $coll(\omega, <\kappa)$. Then in $V[G]$, every set of reals from $L(R)$ is Lebesgue measurable, has the Baire property and has the perfect set property* [Sol70]. Motivated by the stationary tower and [FMS88], Shelah and Woodin showed that the existence of a supercompact cardinal implies that every set of reals from $L(R)$ is Lebesgue measurable and has the Baire property [SW90]. Later, using the stationary tower Woodin proved the same result assuming the existance of a Woodin cardinal limit of woodin cardinals instead of the existance of a supercompact cardinal.

Using the stationary tower, Woodin also proved the so called derived model theorem.

The stationary tower forcing is a very useful and powerful method. This is due to the different kinds of generic elementary embeddings that can be obtained using it. In the following chapters, the reader will be guided through the main results, the definitions and the different constructions of the stationary tower in order to understand the applications of the method presented in the last chapter.

The definition of a stationary set over a set X is given using closed and unbounded sets of X. We will work with two definitions of closed and unbounded sets. In order to show that there is no mistake or ambiguity on the definition of a stationary set, the first step is to study the relations between these different definitions and show that the definition of a stationary set does not depend on the definition of a closed and unbounded set. This makes the argument on some proofs easier, choosing the appropiate definition. But the definition of a stationary set still depends on X, in order to find a relation between the stationary sets over X and the stationary sets over Y, when X is a subset of Y, we define the projection and lifting operators. These two operators have a lot of nice properties that make the work with stationary sets easier.

Now that we have tools to work with stationary sets we can define the stationary tower as the set with all the elements of V_κ that are stationary, and the order given by lifting ($a \geq b$, if the projection of b to $\cup a$ is a subset of a). At this moment every property of the stationary tower will depend on the properties of lifting and projection. A way to obtain results that not only works for the stationary tower is restricting the stationary tower to suborders with equivalent operators for projection and lifting. These suborders are the towers $\mathbb{Q}^S_{<\kappa}$ that are defined using a stationary set S for everyone.

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The tower $\mathbb{Q}^S_{<\kappa}$ is the suborder of $\mathbb{P}_{<\kappa}$ where the conditions are only the subsets of S, the the properties of $\mathbb{Q}^S_{<\kappa}$ depend exclusively on the stationary set S ($\mathbb{Q}^S_{<\kappa}$ is closed under projection if and only if for every subset Y of V_κ, the restriction of S to Y coincides with S_Y). For every generic G and every set X we construct an ultrafilter U_X, given a family of ultrapowers, this family of ultrapowers has a direct limit, the generic ultrapower (M, E) and an elementary embedding j associated to it, this one is the generic elementary embedding and it is the embedding that is used in all the applications. As can be expected, some properties of j depend on S, (when S is of the form $\mathcal{P}_\lambda(V_\kappa)$ the image of λ is at least κ).

Even when S is of the form $\mathcal{P}_\lambda(V_\kappa)$ there are still some general properties vary from one tower to another one. Some of these properties can be studied through the properties of κ (if κ is a limit of completely Jónsson cardinals then $j(\lambda) = \kappa$). Another important property of M that has not been study at this point is the wellfoundedness of M, to study this we introduce the definition of a semi-proper set. This definition is the main idea behind the study of the wellfoundedness of M but also it is very usefull at the moment we want to use the stationary tower forcing method.

At this point the reader should be acquainted with the stationary tower forcing and have the background to understand the classical applications. We present Woodin proof of the absolutness of the theory of the Chang model $L(Ord^\omega)$ under all set forcings from a proper class of Woodin cardinals. To prove this we use symmetric extensions and the tower $\mathbb{Q}^S_{<\kappa}$ when S is the set $\mathcal{P}_{\omega_1}(V_\kappa)$, this tower has nice properties, it makes every element of V_κ countable and if G is a $\mathbb{Q}^S_{<\kappa}$-generic then $G \cap \mathbb{Q}^S_{<\delta}$ is a $\mathbb{Q}^S_{<\delta}$-generic if δ is a Woodin cardinal smaller than κ. As an application we prove that every set of reals from $L(R)$ is Lebesgue measurable, has the Baire property and has the perfect set property if there exists a Woodin cardinal limit of Woodin

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cardinals.

We also prove Σ^2_1-absolutness for forcing with the coutable stationary tower $\mathbb{Q}_{<\delta}$ over models of CH.

Another application shown here is the relation between the stationary towers and the forcing axioms. The forcing axiom $\mathrm{FA}_\alpha(\mathbb{P})$ is the statement that for all sequence $(D_\beta)_{\beta<\alpha}$ of predense sets, there is a filter that intersects every set of the collection. For example Martin maximum is $\mathrm{FA}_{\omega_1}(\mathbb{P})$ for all stationary set preserving forcing $\mathbb{P}$. This application requires knowing the critical point of the generic embedding, this can be done by ensuring that $\{z \in \mathcal{P}_\lambda(\mu) : z \cap \mu \in \mu\} \in G$. Assuming that there is a proper class of Woodin cardinals, we present the proof that the forcing axiom $\mathrm{FA}_\alpha(\mathbb{P})$ holds if and only if $\mathbb{P}$ completely embeds into the stationary tower $\mathbb{Q}^S_{<\delta}$ for $S = \mathcal{P}_\lambda(V_\delta)$ and $\lambda = \alpha^+$ below some condition.

This has applications in Viale's recent work [Via] for proving absoluteness of the Π_2-theory of the H_{ω_2} under stationary set preserving forcings which presented a version of Martin's Maximum.

The books [Lar04] and [Woo99] are good references for further properties and applications of the stationary tower.

None of the uncredited results presented in the preliminaries is due to the author, these results can be found in books of set theory, [Jec03], [Kan03] or in [Kun11]. The uncredited results presented in the chapters 3, 4 and 5 are due to W.H. Woodin.

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Chapter 2

Preliminaries

" *The more I think about language, the more it amazes me that people ever understand each other at all.*

- Kurt Gödel "

2.1 Clubs and stationary sets

There are different notions of closed and unbounded sets (clubs). The aim of this section is to show these different notions and the relation between them. Due to the lemma 2.8 the definition of a stationary set does not depend on the definition of clubs. This brings flexibility in the arguments of the proofs. The next definition is for clubs in $\mathcal{P}(X)$ and is the one that we are going to use in the following chapters.

Definition 2.1. *For a set $X \neq \emptyset$, we say that $C \subseteq \mathcal{P}(X)$ is a club in $\mathcal{P}(X)$, if there exists $F : [X]^{<\omega} \to X$ such that $C = \{Z \subset X : F[[Z]^{<\omega}] \subseteq Z\}$.*
Lets denote by C_f the club associated to the function f.

Definition 2.2. *A set $S \subseteq \mathcal{P}(X)$ is stationary in $\mathcal{P}(X)$, if S intersects all the clubs of $\mathcal{P}(X)$.*

If S is stationary in $\mathcal{P}(X)$ then $\cup S = X$. To check this take $x \in X$, define $F : [X]^{<\omega} \to X$, $F(Z) = x$, then $x \in \bigcup(C_F \cap S)$.

For every set $X \neq \emptyset$, fix an element x in X, the set $S = \{X \backslash \{x\}, \{x\}\}$ is not stationary but $\cup S = X$.
The notion of closed and unbounded set in an ordinal is more natural than the previous one.

Definition 2.3. *Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a club in κ if for every limit ordinal $\alpha < \kappa$ $sup(C \cap \alpha) = \alpha$ implies $\alpha \in C$ and for every $\alpha < \kappa$ there exists $\beta \in C$ such that $\alpha < \beta$.*

These clubs have a property with functions from κ to κ that looks similar to the definition 1.1.

Lemma 2.4. *Let C be a club in κ, then there exists a function $f : \kappa \to \kappa$ such that*

$$\{\gamma < \kappa : \forall \alpha < \gamma,\ f(\alpha) < \gamma\} \subseteq C$$

Proof. Let C be a club in κ. Denote by C' the set of limit points of C, C' is a club in κ. Let $f : \kappa \to \kappa$, $f(\alpha) = \min(C \backslash (\alpha + 1))$, and
$d_f = \{\gamma < \kappa : \forall \alpha < \gamma,\ f(\alpha) < \gamma\}$.

Claim: $d_f = C'$
Proof of the claim: Assume there exists $\gamma \in d_f \backslash C'$, then there exists $\alpha_1, \alpha_2 \in C$ such that $\alpha_1 < \gamma < \alpha_2$ and $C \cap (\alpha_2 \backslash \alpha_1) \subseteq \{\gamma\}$. Since $\forall \alpha < \gamma$ implies $f(\alpha) < \gamma$ we get $\gamma \leq \min(C \backslash \alpha_1 + 1) = f(\alpha_1) < \gamma$ a contradiction, we conclude $d_f \subseteq C'$.
Let $\gamma \in C'$ and $\alpha < \gamma$ then there exists $\beta_1, \beta_2 \in C$ such that $\alpha < \beta_1 < \beta_2 < \gamma$ so $f(\alpha) = \min(C \backslash \alpha + 1) \leq \beta_2 < \gamma$, hence we conclude $C' \subseteq d_f$. □

This notion of clubs in ordinals can be generalize to clubs in $\mathcal{P}_\kappa(X)$ in a natural way as follows.

Definition 2.5. *Let κ be a regular uncountable cardinal. Let X be a set of cardinality at least κ. A set $C \subseteq \mathcal{P}_\kappa(X)$ is a club in $\mathcal{P}_\kappa(X)$ if:*

I. *For every $Z \in \mathcal{P}_\kappa(X)$ exists $Z' \in C$ such that $Z \subseteq Z'$.*

II. *For every chain $x_0 \subset x_1 \subset \cdots$, of length $\alpha < \kappa$, with $x_\gamma \in C$, for every $\gamma < \alpha$ the set $\bigcup_{\gamma<\alpha} x_\gamma$ is in C.*

On the other hand the condition **II.** is equivalent to a condition on directed sets.
A set Y is a directed set if for every x, y in Y there exists $z \in Y$ such that $x \cup y \subseteq z$. We say that a set A is closed for directed sets up to μ if it satisfies the following condition, **II'.** For every directed set $\{z_\gamma : \gamma < \mu\} \subseteq A$ we have $\bigcup_{\gamma<\mu} z_\gamma \in A$.

Lemma 2.6. *A set $C \subseteq \mathcal{P}_\kappa(X)$ satisfies **II** if and only if it satisfies **II'** for every $\mu < \kappa$.*

Proof. $\Leftarrow$: It is clear that every chain is a directed set.
$\Rightarrow$: Proceed by induction on μ. Let Y be a directed set, $Y = \{y_\alpha : \alpha < \mu\}$ and assume C is closed for directed sets up to λ, for every $\lambda < \mu$. Let $Y_\alpha \subseteq Y$ be the smallest set such that $y_\alpha \in Y_\alpha$, $\bigcup_{\beta<\alpha} Y_\beta \in Y_\alpha$ and is a directed set. Then $x_\alpha = \bigcup Y_\alpha \in C$ and $x_\beta \subset x_\alpha$ for $\beta < \alpha$, so $\bigcup Y = \bigcup_{\alpha<\mu} x_\alpha \in C$. □

It is easy to see that every club in $\mathcal{P}(X)$ is unbounded as in **I** and closed as in **II** but has elements of cardinality $|X|$. The following lemma is the version of the lemma 2.4 for clubs in $\mathcal{P}_\kappa(X)$.

Lemma 2.7. *For every club C in $\mathcal{P}_\kappa(X)$ there exists a function $f : [X]^{<\omega} \to \mathcal{P}_\kappa(X)$ such that $\{Z \in \mathcal{P}_\kappa(X) : \forall a \in [Z]^{<\omega}\ f(a) \subseteq Z\} \subseteq C$.*

Proof. Let us construct f in an inductive way.

- $f(\emptyset) = \emptyset$.
- for every $a \in [X]^{n+1}$ choose $f(a)$ such that:
  1. $\forall b \in [a]^n\ f(b) \subseteq f(a)$.

2. $a \subseteq f(a)$.

3. $f(a) \in C$.

This can be done because $a \cup \left(\bigcup_{b \in [a]^n} f(b) \right) \in \mathcal{P}_\kappa(X)$ and C satisfies **I**.

Claim: If $Z \in \mathcal{P}_\kappa(X)$ such that $\forall a \in [Z]^{<\omega}$, $f(a) \subseteq Z$, then
$Z = \bigcup\{f(a) : a \in [Z]^{<\omega}\}$ and $\{f(a) : a \in [Z]^{<\omega}\}$ is a directed subset of C.
With this claim and the lemma 2.6, we get $Z \in C$ and we are done.
Proof of the claim: $\supseteq$: It is clear from the way we chose Z.
$\subseteq$: Let $a \in Z$ then by 2. $\{a\} \subseteq f(\{a\})$.
Directed subset: By 3. $\{f(a) : a \in [Z]^{<\omega}\}$ is a subset of C. Given
$a_1, a_2 \in [Z]^{<\omega}$ there exist n and b such that $b \in [Z]^n$ and $a_1, a_2 \subseteq b$, by 1.
$f(a_1), f(a_2) \subseteq f(b)$. □

Example 2.1. *For κ a regular uncountable cardinal. The set $\{V_\kappa\}$ is not a club in $\mathcal{P}(V_\kappa)$, otherwise there exists $F : [V_\kappa]^{<\omega} \to V_\kappa$, $C_F = \{V_\kappa\}$ and the closure of $\emptyset$ under F is $\{V_\kappa\}$ but $cf(\kappa) > \omega$.*

Some results are easier to prove using tuples than finite subsets, let us define a club of tuples in $\mathcal{P}(X)$, for $|X| \geq \omega$, as the set of closure point of a function.
For a given a function $f : X^{<\omega} \to X$ the associated club is
$C_f = \{a \subseteq X : f[a^{<\omega}] \subseteq a\}$.
Notice that for every function $F : [X]^{<\omega} \to X$ we can define $f : X^{<\omega} \to X$ as $f(Z) = F(Z)$ and we get $C_f = C_F$.

Lemma 2.8. *For every function $f : X^{<\omega} \to X$ there exists a function $F : [X]^{<\omega} \to X$ with $C_F \subseteq C_f$.*

Proof. First let us prove it for the case where X is an ordinal of the form $\delta + \omega$.
Let f be a function $f : X^{<\omega} \to X$. Construct F as follows $F(\emptyset) = \delta$,

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$F(\{\alpha\}) = \alpha + 1$ for all $\alpha \geq \delta$.
For any $a \in [X]^{<\omega}$ let e_a denote the order of a.
Let t_m be $t_m : k_m \to n_m$ be a numeration with $k_m \leq n_m$ such that for all $k \in \omega$ and $t : k \to \omega$ there exists $m \in \omega$ with $k \leq n_m$, $\text{rang}(f) \leq n_m$ and $t_m = t$.
Define $F(a) = f(e_a \circ t_{|a|})$ for $a \notin \{\{\alpha\}, \emptyset\}; \alpha \geq \delta$. Suppose $Z \in C_F$, note that $\alpha \in Z$ for all $\delta \leq \alpha < \delta + \omega$. Let $(a_0, \ldots, a_{k-1}) \in Z^{<\omega}$, $\{a_0, \ldots, a_{k-1}\} = \{b_0, \ldots, b_l\}$, in order, $t : k \to l$ such that $a_i = b_{t(i)}$ for all $i < k$. Pick m such that $n_m \geq l$, $t = t_m$ and $b_{l+1} = \delta' + 1, b_{l+1} = \delta' + 2, \ldots, b_{n_m - 1} = \delta' + (n_m - 1) - (l+1)$ where $\delta' \geq \delta$ such that $\delta' > b_l$.
Then $F(\{b_i : i < n_m\}) = f(e_{b_i : i < n_m} \circ t_m) = f((a_0, \ldots, a_{k-1}))$. Since $\alpha \geq \delta$, $\alpha \in Z$, $\{b_i : i < n_m\} \subset Z$, $f(\{b_i : i < n_m\}) \in Z$.
In the general case of ordinals, $X = \gamma$ and $f : X^{<\omega} \to X$. Let $f' : \gamma + \omega^{<\omega} \to \gamma + \omega$ be the function defined as $f'(a) = f(\{x \in a \cap \gamma\})$, clearly for every $z \in C_{f'}$ we have $z \cap \gamma \in C_f$. By the previous case there exists $F' : [\gamma + \omega]^{<\omega} \to \gamma + \omega$ such that $C_{F'} \subseteq C_{f'}$, since $rang(f') \subseteq \gamma$ we can define $F : [\gamma]^{<\omega} \to \gamma$ as the function $F(x) = F'(x)$ and we will obtain $C_F \subseteq C_f$.
In the general case f induces a function $f_{|X|} : |X|^{<\omega} \to |X|$, $F_{|X|}$ the corresponding function $F_{|X|} : [|X|]^{<\omega} \to |X|$ and let g be a bijection $g : X \to |X|$, define $F(Z) = g^{-1}(F_{|X|}(g[Z]))$. □

2.2 Properties of stationary sets and clubs

In this section we will prove some basic properties of the stationary sets and some others of the club that are fundamental for proofs later.
The clubs described by the next lemma will be very useful when we prove the properties of the stationary tower.

Lemma 2.9. *In a countable language:*

I. *For countably many relations R_i and functions f_i on X, there is a skolem function $F : X^\omega \to X$ such that $M \prec X$ for all $M \in C_F$.*

II. If $F : X^{<\omega} \to X$ is given, then for every $(M, G) \prec (X, F)$ M is closed under F.

Proof. **I.** Let $\{\psi_n : n \in \omega\}$ be an enumeration of all formulas, k_n denote the number of free variables of ψ_n. Let f_{ψ_n} be a skolem function for $\exists y \psi_n(x_1, \ldots, x_{k_n-1}, y)$. Choose $x_0 \in X$. Define $F(a_1, \ldots, a_n) = f_{\psi_n}(a_1, \ldots, a_{k_n-1})$ if $k_n \leq n$ and x_0 in other case.
Then for $M \in C_F$, $\psi_n, a_1, \ldots, a_{k_n-1} \in M$ we have $F(a_1, \ldots, a_{k_n}, a_{k_1+1}, \ldots, a_n) \in M$. If $k_n \leq n$ then $f_{\psi_n}(a_1, \ldots, a_{k_n}) \in M$, so for $k_n \leq n+1$, M is closed for f_{ψ_n}.
It is enough to show that there exists an enumeration $\{\psi_n : n \in \omega\}$ such that $k_n \leq n+1$. Let A_n be the set of all the formulas with n free variables, and $\{\psi_{n,m} : m \in \omega\}$ an enumeration of A_n, then the usual enumeration of $\mathbb{N} \times \mathbb{N}$ works.
II. X is closed under F so M is closed under G, since $(M, G) \prec (X, F)$, then M is closed under F. □

From now on a set a is stationary if it is stationary in $\mathcal{P}(\cup a)$, unless specified differently. Now we will define the projection and the lifting, these two operations will lead us to the construction of stationary towers.

Definition 2.10. *Let $\emptyset \neq X \subseteq Y$.*

- *Projection*
  For $S \subseteq \mathcal{P}(Y)$, we define the projection of S to X as
  $S_X = \{Z \cap X : Z \in S\}$.

- *Lifting*
  For $S \subseteq \mathcal{P}(X)$, we define the lifting of S to Y as
  $S^Y = \{Z \in \mathcal{P}(Y) : Z \cap X \in S\}$.

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Remark. For every $\emptyset \neq X \subseteq Y$ we have the following properties of lifting and projection.

- $(S^Y)_X = S$.
- $S \subseteq (S_X)^Y$.
- $S_1 \subset S_2$ implies $(S_1)_X \subseteq (S_2)_X$.
- $S_1 \subset S_2$ implies $(S_1)^Y \subseteq (S_2)^Y$.
- $(S_1 \cap S_2)_X \subseteq (S_1)_X \cap (S_2)_X$.
- $(S_1 \cap S_2)^Y = (S_1)^Y \cap (S_2)^Y$.
- $(S_1 \cup S_2)_X = (S_1)_X \cup (S_2)_X$.
- $(S_1 \cup S_2)^Y = (S_1)^Y \cup (S_2)^Y$.
- $(\mathcal{P}(X) \backslash S)^Y = \mathcal{P}(Y) \backslash S^Y$.
- $(S_0)_X \subseteq S_1$ if and only if $S_0 \subseteq (S_1)^Y$.

Theorem 2.11 (Menas). *For every $\emptyset \neq X \subseteq Y$ we have:*

1. *If S is a stationary set in $\mathcal{P}(Y)$ then S_X is stationary.*
2. *S is a stationary set in $\mathcal{P}(X)$ if and only if S^Y is stationary.*

Proof. 1. Let $f : [X]^{<\omega} \to X$, define $F : [Y]^{<\omega} \to Y$, $F(\{y_1, \ldots, y_n\}) = f(\{y_1, \ldots, y_n\} \cap X)$. So $Z \subseteq Y$ is closed under F if and only if $Z \cap X$ is closed under f, we get $C_f = (C_F)_X$ by the previous remark we get $(C_f)^Y \supseteq C_F$, since $C_F \cap S \neq \emptyset$.

2. $\Leftarrow$: If S^Y is stationary then by 1. $S = (S^Y)_X$ is stationary.
$\Rightarrow$: Let $F : Y^{<\omega} \to Y$, given a $Z \subseteq Y$ define $Z_0 = Z$, $Z_{i+1} = F[Z_i^{<\omega}] \cup Z_i$ and $H_F(Z) = \bigcup_{i \in \omega} Z_i$. For the language $\mathcal{L} = \{g_i : i \in \omega\} \cup \{x_i : i \in \omega\}$ where g_i is an i-ary function and $\{x_i\}_{i \in \omega}$ are variables. Let $\{t_m\}_{m \in \omega}$

be an enumeration of all the terms that are not variables, such that all terms appear infinitely often. Fix $x_0 \in H_F \cap X$ then define $g : X^{<\omega} \to X$ as $g(a_0, \ldots, a_m) = t_m(a_i \backslash x_i\ ,\ F \backslash g_i)$ if this belongs to X, x_0 in other case. So $g[Z^{<\omega}] = H_F(Z) \cap X$. Assume $Z \in C_g$, $H_F(Z) \cap X \subseteq Z$, since $Z_0 = Z$ then $H_F(Z) \cap X = Z$ and $H_F(Z) \in C_F$ we conclude $C_g \subseteq (C_F)_X$. Since S is stationary, $C_g \cap S \neq \emptyset$ let $Z \in C_g \cap S$, then $Z \in (C_F)_X \cap S$ so there exists $Z' \in C_F$ such that $Z = Z' \cap X$ so $Z' \in S^Y$, $C_F \cap S^Y \neq \emptyset$. □

Note that for every function $F : X^{<\omega} \to X$ and $Z = \{\emptyset\}$, $H_F(Z)$ is countable, so $\mathcal{P}_\lambda(X)$ is stationary for $\omega < \lambda$.

Remark. There exists a no stationary set $S \subseteq \mathcal{P}(Y)$, such that S_X is stationary. Let $X = \{x_1, x_2\}$, $S = \{\{x_1\}, \{x_2\}\}$, clearly S is not stationary in $\mathcal{P}(X)$, for the function $F(x) = x_1$ for $x \in \{\{x_2\}, \emptyset\}$ and $F(x) = x_2$ in the other case, $C_F \cap S = \emptyset$; but $S_{\{x_1\}}$ is stationary in $\mathcal{P}(\{x_1\})$.

The following lemma will be used in the following chapter as the **normality** lemma.

Lemma 2.12 (Jech). *Let $X \neq \emptyset$ and $S \subseteq \mathcal{P}(X)$ stationary. Suppose $F : S \to X$ is a regresive function, i.e. $F(Z) \in Z$ for all $Z \in S$. Then there exists $a \in X$ such that $\{Z \in S : F(Z) = a\}$ is stationary.*

Proof. Assume that for each $a \in X$ there exists $F_a : X^{<\omega} \to X$ such that $C_{F_a} \cap \{Z \in S : F(Z) = a\} = \emptyset$. Define $G : X^{<\omega} \to X$ as
$G(a, a_0, \ldots a_n) = F_a(a_0, \ldots, a_n)$.
Let $Z \in C_G \cap S$ and $x \in Z$, $(a_0, \ldots, a_n) \in Z^{<\omega}$ then
$F_x(a_0, \ldots, a_n) = G(x, a_0, \ldots, a_n) \in Z$, thus $Z \in C_{F_x}$ so $F(Z) \neq x$ for all $x \in Z$, $F(Z) \notin Z$ a contradiction. □

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To conclude this section we will show what it is known as the club filter that is really useful at the moment you work with clubs. This filter is really different when the clubs are in $\mathcal{P}_\kappa(X)$ than when the clubs are in $\mathcal{P}(X)$.

Lemma 2.13 (Jech). *If C and D are clubs in $\mathcal{P}_\kappa(X)$, then $C \cap D$ is a club. All the clubs in $\mathcal{P}_\kappa(X)$ generate a κ-complete filter.*

Proof. For the first part.

II: Let C and D be clubs. Every chain $x_1 \subseteq x_2 \subseteq \cdots$ with length $\gamma < \kappa$ in $C \cap D$ is a chain in C and D so the limit point $\bigcup_{\alpha<\gamma} x_\alpha$ is in C and D so it is in $C \cap D$.

I: Let $x \in \mathcal{P}_\kappa(X)$, choose $x_0 \in C$ such that $x \subseteq x_0$, $x_{2i} \in C$ and $x_{2i-1} \in D$, $i > 0$ such that $x_{i-1} \subseteq x_i$. This chain has the same limit as the chains $< x_{2i} >_{i\geq 0}$ and $< x_{2i-1} >_{i>0}$, we conclude $x \subseteq \bigcup_{i\in\omega} x_i \in C \cap D$.

For the κ-completeness we proceed by induction, for the successor step this is the same proof as in the beginning of this lemma, let $\gamma < \kappa$ be a limit ordinal, it is enough to prove that $C = \bigcap_{\alpha<\gamma} C_\alpha$, where $< C_\alpha >_{\alpha<\gamma}$ is a decreasing chain, is a club.

II: Let $x_1 \subseteq x_1 \subseteq \cdots$ be a chain of length $\alpha < \gamma$ in C, for every $\alpha < \beta < \gamma$ this chain is a chain in C_β then the limit is in C_β.

I: Given $x \in \mathcal{P}_\kappa(X)$ let $x_0 \in C_0$ such that $x \subseteq x_0$ and $x_\alpha \in C_\alpha$ such that $\bigcup_{\beta<\alpha} x_\beta \subseteq x_\alpha$, for every α the chain $< x_\beta >_{\alpha\leq\beta<\gamma}$ is an increasing chain in C_α, so $\bigcup_{\beta<\alpha} x_\beta \in C_\alpha$ then $\bigcup_{\beta<\alpha} x_\beta \in C$. □

Lemma 2.14. *If C_f and C_g are clubs in $\mathcal{P}(X)$, then $C_f \cap C_g$ is a club in $\mathcal{P}(X)$,.*

Proof. Let $f : X^{<\omega} \to X$ and $g : X^{<\omega} \to X$ be the associated function to the clubs. Define $F : X^{<\omega} \to X$ as $F(a_0, a_1, \ldots, a_n) = f(a_0, a_1, \ldots a_k)$ when $n = 2k$ and $F(a_0, a_1, \ldots, a_n) = g(a_0, a_1, \ldots a_k)$ when $n = 2k + 1$.

Assume $Z \in C_f \cap C_g$ and $a = (a_0, a_1, \ldots, a_n) \in Z^{<\omega}$, then $F(a) = f(a_0, a_1, \ldots a_k)$ or $F(a) = g(a_0, a_1, \ldots a_k)$, in both cases $F(a) \in Z$ and $C_f \cap C_g \subseteq C_F$. Assume $Z \in C_F$ and $a = (a_0, a_1, \ldots, a_n) \in Z^{<\omega}$, then $f(a) = F(a_0, a_1, \ldots, a_n, a_{n+1}, \ldots a_{2n})$, where $a_{n+1}, \dots a_{2n} \in Z$, then $Z \in C_f$, in the same way we prove $Z \in C_g$, we conclude $C_F = C_f \cap C_g$. □

Lemma 2.15 (Foreman, Magidor, Shelah). *For $|X| \geq \omega$, the collection of clubs in $\mathcal{P}(X)$ generates a countably complete filter.*

Proof. Let $< f_i >_{i\in\omega}$ a set of functions from $X^{<\omega}$ to X. Let $g : \omega \to \omega \times \omega$ a surjective function such that $g_2(n) \leq n$.
Define $F : X^{<\omega} \to X$ as $F(a_0, a_1, \dots, a_n) = f_{g_1(n)}(a_0, a_1, \dots, a_{g_2(n)})$.
Let $Z \in \bigcap_{i\in\omega} C_{f_i}$ and $a = (a_0, a_1, \dots, a_n) \in Z^{<\omega}$, then $F(a) = f_{g_1(n)}(a_0, a_1, \dots, a_{g_2(n)})$ so $F(a) \in Z$, therefore $\bigcap_{i\in\omega} C_{f_i} \subseteq C_F$. Let $Z \in C_F$ and
$a = (a_0, a_1, \dots, a_n) \in Z^{<\omega}$, then for every $i \in \omega$ there exists k_i such that $g_1(k_i) = i$ and $g_2(k_i) = n$, and computing $f_i(a) = F(a_0, a_1, \dots, a_n, a_{n+1}, \dots a_{k_i})$, where $a_{n+1}, \dots a_{k_i} \in Z$, then for every $i \in \omega$, $Z \in C_{f_i}$, $C_F = \bigcap_{i\in\omega} C_{f_i}$. □

At this moment we have two different kinds of clubs, definition 2.1 and 2.5, as we said, any club C in $\mathcal{P}(V_\kappa)$ has V_κ as an element so C is not a club in $\mathcal{P}_\kappa(V_\kappa)$. These two kinds of clubs are related when X is a regular uncountable cardinal λ and we restrict the definition 2.1 as, the clubs $C \subseteq \mathcal{P}_\kappa(\lambda)$ such that there exists $F : [\lambda]^{<\omega} \to \lambda$ with $C = \{Z \subset X : F[[Z]^{<\omega}] \subseteq Z \wedge |Z| < \kappa\}$, we call them strong clubs. The following lemma due to Foreman, Magidor, Shelah shows this relation and it can be found in [FMS88].

Lemma 2.16 (Foreman, Magidor, Shelah). *Let $\kappa < \lambda$ be regular cardinals, $\mathcal{F}_\mathcal{S}(\lambda, \kappa)$ the filter of strong clubs in $\mathcal{P}_\kappa(\lambda)$ and $\mathcal{F}(\lambda, \kappa)$ the filter of clubs in $\mathcal{P}_\kappa(\lambda)$. Then $\mathcal{F}(\lambda, \kappa)$ is the filter generated by*

$$\mathcal{F}_\mathcal{S}(\lambda, \kappa) \cup \{\{Z \in \mathcal{P}_\kappa(\lambda) : Z \cap \kappa \in \kappa\}\}$$

Proof. Let $C \subset \mathcal{P}_\kappa(\lambda)$ be a club, $\mathcal{L} =< H(\lambda), \varepsilon, C, \Delta, \{\kappa\} >$, and
$\{g_i\}_{i\in\omega}$ Skolem functions for $\mathcal{L}$ closed under composition. Let $\{f_i\}_{i\in\omega}$ their restriction to λ in domain and range. For any set $y \in \mathcal{P}_\kappa(\lambda)$ that is closed under each f_i and satisfies $y \cap \kappa \in \kappa$, exists $N \prec \mathcal{L}$ such that $N \cap \lambda = y$.

For $\alpha \in [y]^{<\omega}$ define M_α as follows, for $|\alpha| = 1$ let $M_\alpha \in N \cap C$ be such that $\alpha \subset M_\alpha$, assume M_α is define for every $|\alpha| < n$ and let $\beta \in [y]^{<\omega}$ be a set with n+1 elements, since $N \models$"C is a club" and $(\bigcup_{\alpha \subset \beta} M_\alpha) \cup \beta \in N$ there exists $M_\beta \in N \cap C$ such that $(\bigcup_{\alpha \subset \beta} M_\alpha) \cup \beta \subset M_\beta$.
For every α, since $M_\alpha \in C \cap N$, then $N \models |M_\alpha| < \kappa$. Being $h : \gamma \to M_\alpha$ a surjective function in N, where $\gamma = |M_\alpha|$ in N; since $N \cap \kappa \in \kappa$, we get $\gamma \in N$ and for every $x \in M_\alpha$ there exists $\beta \in N$ such that $f(\beta) = x \in N$. Therefore $M_\alpha \in N$.
Since $\{M_\alpha : \alpha \in [y]^{<\omega}\}$ is a directed set and $y = \bigcup M_\alpha$, we conclude $y \in C$. By lemma 2.15 exists a function f such that $C_f = \cap C_{f_i}$, therefore $\{y \in \mathcal{P}_\kappa(\lambda) : y \in C_f \wedge y \cap \kappa \in \kappa\} \subset C$. □

If in the definition 2.5 (I) we allow $|Z| = \kappa$, then X is an element of any club of this kind, and $\{X\}$ is a club, and $\mathcal{F}(\kappa, \kappa)$ would be the principal filter generated by $\{X\}$ and example 2.1 shows that this new kind of club is different from the one in definition 2.1.

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Chapter 3

The stationary towers

" *The mind, the head is the territory where nothing should be banned.*

- Los Rodríguez, Aquí No Podemos Hacerlo "

3.1 The stationary tower and its restrictions

For κ a strongly inaccessible cardinal, the stationary tower in V_κ is denoted by $\mathbb{P}_{<\kappa}$ and is the set whose elements are all the $a \in V_\kappa$, that are stationary, and we say $a \geq b$ if $\cup a \subseteq \cup b$ and for every $z \in b$, $z \cap (\cup a) \in a$, $(b \subseteq a^{\cup b})$, note that two conditions a and b are compatible if $a^{(\cup a)\cup(\cup b)} \cap b^{(\cup a)\cup(\cup b)}$ is stationary in $(\cup a) \cup (\cup b)$.
There are many different ways to restrict the stationary tower to a suborder, the following one is due to W. Hugh Woodin, it can be found in [Lar04].

Definition 3.1 ($\mathbb{P}^S_{<\kappa}$). *For κ a strongly inaccessible cardinal and $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ a stationary set, for every $\lambda < \kappa$ let $S_\lambda = \{X \cap V_\lambda : X \in S\}$ and*

$$\mathbb{P}^S_{<\kappa} = \{a \in \mathbb{P}_{<\kappa} : a \subseteq S_{sup((\cup a)\cap\kappa)}\}$$

with the induced order.

The following remark motivates the definition of the towers $\mathbb{Q}^S_{<\kappa}$, and will be the suborders with in we are going to work, as we will see, sometimes

$\mathbb{P}^S_{<\kappa}$ is dense in $\mathbb{Q}^S_{<\kappa}$ and the stationary tower can be seen as a tower $\mathbb{Q}^S_{<\kappa}$ with the precise set S.

Remark. The usual projection doesn't hold for $\mathbb{P}^S_{<\kappa}$, when $S = \mathcal{P}_{\omega_1}(V_\kappa)$. Take $b = \mathcal{P}_{\omega_1}(V_\lambda) \cap V_\lambda$ for λ an uncountable limit cardinal. Thus if $x \in b$ we get $x \in V_\lambda$, $x \in \mathcal{P}_{\omega_1}(V_\kappa)$ and $sup((\cup b) \cap \kappa) = \lambda$, $S_\lambda = \{Z \cap V_\lambda : Z \in \mathcal{P}_{\omega_1}(V_\kappa)\}$, so $b \in \mathbb{P}^S_{<\kappa}$. Now let $X = \{\omega\}$, then $b_X = \{Z \cap X : Z \in b\} = \{\{\omega\}, \emptyset\}$, $sup((\cup b_X) \cap \kappa) = \omega$. Since $\{\omega\} \notin V_\omega$, we conclude $\{\omega\} \notin S_\omega$ and $b_X \notin \mathbb{P}^S_{<\kappa}$.
Even for $\cup a, X$ transitive sets, the projection still doesn't hold. Take
$b = \mathcal{P}_{\omega_1}(V_\lambda) \cap V_\lambda$ and $X = \omega + 1$, $b_X = \mathcal{P}(X)$ and $\{\omega\} \notin S_\omega$.
In $\mathbb{P}^S_{<\kappa}$ we can project to $X = V_\lambda$ for λ a limit cardinal. Given $b \in \mathbb{P}^S_{<\kappa}$ and $\theta = sup((\cup b) \cap \kappa)$, then for $\lambda \geq \theta$ we have $b_X = b$ and for $\theta \geq \lambda$, b_X is stationary and $\cup b_X = X$ so $sup((\cup b_X) \cap \kappa) = sup(V_\lambda \cap \kappa) = \lambda$, $S_\lambda = \{Z \cap V_\lambda : Z \in S\}$. Since $b \in \mathbb{P}^S_{<\kappa}$ and $b_X = \{Z \cap V_\lambda : Z \in b\}$, this implies $Z \in b$ then there exists $Z' \in S$ such that $Z = Z' \cap V_\theta$, so $b_X \subseteq S_\lambda$.

Definition 3.2 ($\mathbb{Q}^S_{<\kappa}$). *For κ a strongly inaccessible cardinal and $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ a stationary set, let*

$$\mathbb{Q}^S_{<\kappa} = \{a \in \mathbb{P}_{<\kappa} : a \subseteq S\}$$

with the induced order.

Remark. For $S = \mathcal{P}_\kappa(V_\kappa)$ we get $\mathbb{Q}^S_{<\kappa} = \mathbb{P}_{<\kappa}$. By definition $\mathbb{Q}^S_{<\kappa} \subseteq \mathbb{P}_{<\kappa}$ and by the transitivity of V_κ, $\mathbb{P}_{<\kappa} \subseteq \mathbb{Q}^S_{<\kappa}$.

The last restriction is given by Matteo Viale can be found in [Via].

Definition 3.3 ($\mathbb{R}^\lambda_\kappa$). *For κ a strongly inaccessible cardinal and λ a regular cardinal, let $\mathcal{R}_\lambda = \{X : X \cap \lambda \in \lambda \wedge |X| < \lambda\}$ and*

$$\mathbb{R}^\lambda_\kappa = \{a \in \mathbb{P}_{<\kappa} : a \subset \mathcal{R}_\lambda\}$$

with the induced order.

Before starting with the properties of the $\mathbb{Q}^S_{<\kappa}$ towers let's make an observation about the stationary tower in V_κ.

Lemma 3.4. *For every $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ stationary set, $\mathbb{P}_{<\kappa} \neq \mathbb{P}^S_{<\kappa}$.*

Proof. It is enough to prove this for $S = \mathcal{P}_\kappa(V_\kappa)$. Let $a = \{\{\omega\}\}$, a is stationary in $\mathcal{P}(\cup a)$ and $(\cup a \cap \kappa) = \{\omega\}$ and $S_\omega = \{X \cap V_\omega : X \in \mathcal{P}_\kappa(V\kappa)\}$. Since $\omega \notin V_\omega$ then $\{\omega\} \notin S_\omega$, $a \nsubseteq S_\omega$. □

In the definition of lifting for stationary sets we asked the sets a and Y, to satisfy the property $\cup a \subseteq Y$ to be able to define a^Y. But if $Y \subseteq \cup a$ we can define $a^Y = \{Z \subseteq Y : Z \cap (\cup a) \in a\}$, that turns to be a restriction $a^Y = \{Z \subseteq Y : Z \in a\}$.

Now we define the notion of projection and lifting in $\mathbb{Q}^S_{<\kappa}$, this notion will let us have a nice tower with in we can work easily most of the times, in particular when $S = \mathcal{P}_{\omega_1}(V_\kappa)$ that is an important case because of its applications.

Definition 3.5. *Given $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ a stationary set,*

- *Projection for $\mathbb{Q}^S_{<\kappa}$.*
  For every $a \in \mathbb{Q}^S_{<\kappa}$ and $Y \subseteq \cup a$, define

$$a_Y = \{Z \cap Y : Z \in a\}$$

- *Lifting for $\mathbb{Q}^S_{<\kappa}$.*
  For every $a \in \mathbb{Q}^S_{<\kappa}$ and $\cup a \subseteq Y$, define

$$a^Y = \{Z \in S^Y : Z \cap (\cup a) \in a\}$$

Note that the lifting for $\mathbb{Q}^S_{<\kappa}$ in the case $S = \mathcal{P}_\kappa(V_\kappa)$, is the same lifting as in the definition 2.10. And that the lifting in $\mathbb{Q}^S_{<\kappa}$ is $a^Y = a^Y \cap S^Y$ where, the lifting of the left side is in $\mathbb{Q}^S_{<\kappa}$ and the right side is in $\mathbb{P}_{<\kappa}$.

We will say that $\mathbb{Q}^S_{<\kappa}$ is closed under projection (lifting) if for any $a \in \mathbb{Q}^S_{<\kappa}$

and $Y \subset \cup a$ ($\cup a \subseteq Y$) we have $a_Y \in \mathbb{Q}^S_{<\kappa}$ ($a^Y \in \mathbb{Q}^S_{<\kappa}$). As it was mentioned the towers $\mathbb{Q}^S_{<\kappa}$ are not always closed under projection, but the theorem 3.6 characterized when the tower is closed under projection.

Theorem 3.6. *Let $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ be a stationary set, then $\mathbb{Q}^S_{<\kappa}$ is closed under projection if and only if every subset $Y \subseteq V_\kappa$ satisfies $S_Y = S^Y$.*

Proof. $\Leftarrow$. Let $a \in \mathbb{Q}^S_{<\kappa}$, for every $Y \subseteq \cup a$ we have $a_Y = \{Z \cap Y : Z \in a\}$. So $a_Y \subseteq S_Y = S^Y \subseteq S$, $a \in \mathbb{Q}^S_{<\kappa}$.
$\Rightarrow$: Assume there exists $X \subseteq V_\kappa$ such that $S_X \neq S^X$. Let $x \in S_X \backslash S$ then $x = Z \cap X$ for some $Z \in S$; let $a = \{Z\}$, clearly $a \in \mathbb{Q}^S_{<\kappa}$ and since $\mathbb{Q}^S_{<\kappa}$ is closed under projection, $a_X \in \mathbb{Q}^S_{<\kappa}$, this means $\{x\} \in \mathbb{Q}^S_{<\kappa}$, $x \in S$ contradicting the way we chose x. □

For lifting again not every tower $\mathbb{Q}^S_{<\kappa}$ is closed under lifting but some properties of S ensure that the respective tower will be closed under lifting.
Note that if for every $Y \subseteq V_\kappa$, S^Y contains a club in $\mathcal{P}(Y)$, then $\mathbb{Q}^S_{<\kappa}$ is closed under lifting. The lemma 3.7 gives us a sufficient condition for $\mathbb{Q}^S_{<\kappa}$ to be closed under lifting.

Lemma 3.7. *Let $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ be a stationary set such that for every $Y \subseteq V_\kappa$ and $F : [Y]^{<\omega} \to Y$, S satisfy $H_F(Z) \in S^Y$ for every $Z \in S^Y$. Then $\mathbb{Q}^S_{<\kappa}$ is closed under lifting.*

Proof. Let $a \in \mathbb{Q}^S_{<\kappa}$, $\cup a \subseteq Y$ and $F : [Y]^{<\omega} \to Y$, by the proof of the theorem 2.11, there exists $g : [\cup a]^{<\omega} \to \cup a$ such that $C_g \subseteq (C_F)_{\cup a}$ then $C_g \cap a \neq \emptyset$, let $Z \in C_g \cap a \subseteq S^{\cup a}$, since $\cup a \subseteq Y$ then $H_F(Z) \in S^Y$. But here exists $Z' \in C_F$ such that $Z' \cap (\cup a) = Z$ and $H_F(Z) \subseteq Z'$, so $H_F(Z) \cap (\cup a) = Z$, we conclude $H_F(Z) \in a^Y$. □

Example 3.1. *Let $S = \mathcal{P}_\kappa(V_\kappa) \backslash \mathcal{P}_{\omega_1}(V_\kappa)$, $a = \{\omega_1\} \in \mathbb{Q}^S_{<\kappa}$ and $a_\omega = \{\omega\} \notin \mathbb{Q}^S_{<\kappa}$. S is stationary because for every function $F : [V_\kappa]^{<\omega} \to V_\kappa$, $\omega < H_F(\omega_1) < \kappa$. For each $a \in \mathbb{Q}^S_{<\kappa}$, $|\cup a| > \omega$ and for every $\cup a \subseteq Y$ we have*

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$\forall x \in a^Y$ $\omega_1 \subseteq x$, $|x| > \omega$, so $a^Y \in \mathbb{Q}^S_{<\kappa}$.
$\mathbb{Q}^S_{<\kappa}$ is closed under lifting but not under projection, notice that $S_\omega \neq S^\omega$ and that S^ω doesn't contain a club.

Example 3.2. *Let $x_0, x_1 \in \omega_1 \backslash \omega$, $x_0 \neq x_1$.*

$$S = \{\omega, \omega \cup \{x_0\}, \omega_1 \backslash \{x_0\}\} \cup (\mathcal{P}_\kappa(V_\kappa) \backslash \mathcal{P}_{\omega_2}(V_\kappa))$$

$a = \{\omega\} \in \mathbb{Q}^S_{<\kappa}$, $a^{\omega_1} = \{\omega, \omega \cup \{x_0\}, \omega_1 \backslash \{x_0\}\}$, $\cup a^{\omega_1} = \omega_1$.
Let $F : \omega_1^{<\omega} \to \omega_1$ be the function defined as $F(Z) = x_0$ for $Z \neq x_0$ and $F(x_0) = x_1$. We get $F[(\omega \cup \{x_0\})^{<\omega}] \nsubseteq \omega \cup \{x_0\}$ so $a^{\omega_1} \cap C_F = \emptyset$.

Example 3.3. *The countable tower is denote by $\mathbb{Q}_{<\kappa}$ and it is the tower $\mathbb{Q}^S_{<\kappa}$ when $S = \mathcal{P}_{\omega_1}(V_\kappa)$. Note that $S_X = \mathcal{P}_{\omega_1}(X) = S^X$ for every $X \subseteq V_\kappa$ so $\mathbb{Q}_{<\kappa}$ is closed under projection and lifting. This holds every time that $S = \mathcal{P}_\lambda(V_\kappa)$ for $\omega < \lambda$.*

Example 3.4. *Let λ be a cardinal such that $\omega < \lambda < \kappa$, $S = \mathcal{P}(V_\lambda) \cup \mathcal{P}_\omega(V_\kappa)$ and $a = \mathcal{P}(V_\lambda) \backslash \mathcal{P}_\omega(V_\lambda)$. Clearly $a \in \mathbb{Q}^S_{<\kappa}$ and $S_Y = S^Y$ for every $Y \subseteq V_\kappa$, but $a^V_\delta = a$ for every $\lambda < \delta$, then a^V_δ is not stationary in V_δ. $\mathbb{Q}^S_{<\kappa}$ is closed under projection but not under lifting.*

Lemma 3.8 (Normality in $\mathbb{Q}^S_{<\kappa}$). *Let $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ be a stationary set, $b \in \mathbb{Q}^S_{<\kappa}$ and $F : b \to \cup b$ a regressive function, i.e. $F(Z) \in Z$. Then there exists $x \in \cup b$ such that $\{Z \in b : F(Z) = x\} \in \mathbb{Q}^S_{<\kappa}$.*

Proof. By the normality lemma there exists $x \in \cup b$ such that $\{Z \in b : F(Z) = x\} \in \mathbb{P}_{<\kappa}$ and $\{Z \in b : F(Z) = x\} \subseteq b \subseteq S$. We conclude $\{Z \in b : F(Z) = x\} \in \mathbb{Q}^S_{<\kappa}$. □

Corollary 3.9. *Let $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ be stationary, such that $\mathbb{Q}^S_{<\kappa}$ is closed under projection and lifting, then $\mathbb{P}^S_{<\kappa}$ is dense in $\mathbb{Q}^S_{<\kappa}$.*

Proof. By definition $\mathbb{P}^S_{<\kappa} = \{a \in \mathbb{P}_{<\kappa} : a \subseteq S_{sup((\cup a)\cap\kappa)}\}$ then
$\mathbb{P}^S_{<\kappa} = \{a \in \mathbb{P}_{<\kappa} : a \subseteq S^{V_\lambda},\ \lambda = sup((\cup a) \cap \kappa)\}$, so for every $a \in \mathbb{P}^S_{<\kappa}$ we have $a \subseteq S$ and $a \in \mathbb{Q}^S_{<\kappa}$ proving $\mathbb{P}^S_{<\kappa} \subseteq \mathbb{Q}^S_{<\kappa}$.

Let $a \in \mathbb{Q}^S_{<\kappa}$, so $a \in \mathbb{P}_{<\kappa}$ and there exists α a limit cardinal such that $a \in V_\alpha$, $a^{V_\alpha} \in \mathbb{Q}^S_{<\kappa}$ and $\cup a^{V_\alpha} = V_\alpha$, therefore $sup((\cup a^{V_\alpha}) \cap \kappa) = \alpha$ and $a^{V_\alpha} \subseteq S^{V_\alpha} = S_{V_\alpha} = S_\alpha$, we conclude $a^{V_\alpha} \in \mathbb{P}^S_{<\kappa}$ and $a^{V_\kappa} \leq a$. □

Remark. In the example 3.1 we have $\mathbb{P}^S_{<\kappa} \nsubseteq \mathbb{Q}^S_{<\kappa}$, because $\mathbb{Q}^S_{<\kappa}$ is closed under lifting but not under projection. In the example 3.3 $\mathbb{P}^{\mathcal{P}_{\omega_1}(V_\kappa)}_{<\kappa}$ is dense in $\mathbb{Q}_{<\kappa}$.

Fact 3.10. *Let $S \subseteq \mathcal{P}_\kappa(V_\kappa)$ be stationary, such that $\mathbb{Q}^S_{<\kappa}$ is closed under projection and lifting. Then for every club subset C of κ, there exists a predense set $D_C \subset \mathbb{Q}^S_{<\kappa}$ such that $\gamma \in C$ for every inaccessible γ such that, $D_C \cap \mathbb{Q}^S_{<\gamma}$ is a predense subset.*

Proof. Let $< \gamma_\alpha : \alpha < \kappa >$ be an enumeration of C. For each $a \in \mathbb{P}^S_{<\kappa}$ define $a^* = a^{V_{\gamma_\alpha}}$ where γ_α is the least ordinal in C such that $a \in V_{\gamma_\alpha}$.
Define $D_C = \{a^* : a \in \mathbb{P}^S_{<\kappa}\}$. Note that D_C is dense because $\mathbb{P}^S_{<\kappa}$ is dense. Let γ be such that $D_C \cap \mathbb{Q}^S_{<\gamma}$ is predense, and $< \alpha_\beta : \beta < cof(\gamma) >$ a cofinal succession of γ. For each α_β take $a_\beta \in \mathbb{Q}^S_{<\gamma}$ such that $V_{\alpha_\beta} \subseteq \cup a_\beta$, therefore, for each a_β there exists $b_\beta \in D_C \cap \mathbb{Q}^S_{<\gamma}$ compatible with a_β, by corollary 3.9, there exists $r \in \mathbb{P}^S_{<\gamma}$ such that $r \leq a_\beta$ and $r \leq b_\beta$; since $\mathbb{P}^S_{<\gamma} \subset \mathbb{P}^S_{<\kappa}$, r^* is defined and satisfies $V_{\alpha_\beta} \subseteq \cup r^* = V_{\gamma_\theta}$, for some $\gamma_\theta \geq \alpha_\beta$. We conclude there exists a succession $< \gamma_\beta : \beta < \gamma > \subseteq C$ cofinal to γ, since C is a club $\gamma \in C$. □

3.2 The generic ultrapower

In this section we will construct an ultrapower for the towers $\mathbb{Q}^S_{<\kappa}$ in the same way as W. Hugh Woodin did in $\mathbb{P}_{<\kappa}$.
From now on S is a stationary set in $\mathcal{P}_\kappa(V_\kappa)$, with κ a strongly inaccessible cardinal, and $\mathbb{Q}^S_{<\kappa}$ is closed under projection and lifting, unless we state something different.
Given $G \subset V_\kappa$ a V-generic in $\mathbb{Q}^S_{<\kappa}$, for every nonempty set $X \in V_\kappa$ we define

$$U_X = \{b_X : b \in G \wedge X \subseteq \cup b\}$$

Fact 3.11. *U_X is a V-ultrafilter on S^X.*

Proof. Let $b \in \mathbb{Q}^S_{<\kappa}$, $X \subseteq \cup b$ and $A \subset S^X$.
Define $b^0 = \{Z \in b : Z \cap X \in A\}$ and $b^1 = \{Z \in b : Z \cap X \notin A\}$. Since b is stationary and $b^0 \cup b^1 = b$, b^0 is stationary over $\mathcal{P}(\cup b)$ or b^1 is stationary over $\mathcal{P}(\cup b)$. Assume b^0 is stationary, $\cup b^0 = \cup b$ so $b^0 \subseteq b \subset S$ then $b^0 \in \mathbb{Q}^S_{<\kappa}$ and $b^0 \leq b$, the same arguments show that $b^1 \subseteq b \subset S$, thus $b^1 \in \mathbb{Q}^S_{<\kappa}$ and $b^1 \leq b$.
We have shown that for each $A \subset S^X$ and each $b \in \mathbb{Q}^S_{<\kappa}$ with $X \subseteq \cup b$, exists an element $a \in \mathbb{Q}^S_{<\kappa}$ such that $a_X \subseteq A$ or $a_X \subseteq S^X \backslash A$. Then the set

$$D = \{a \in \mathbb{Q}^S_{<\kappa} : a_X \subseteq A \vee a_X \subseteq S^X \backslash A\}$$

is dense in $\mathbb{Q}^S_{<\kappa}$. Since G is generic, $G \cap D \neq \emptyset$ so there exists $b_X \in G$ such that $b_X \leq A$ or $b_X \leq S^X \backslash A$, $A \in G$ or $S^X \backslash A \in G$, we conclude $A \in U_X$ or $S^X \backslash A \in U_X$. □

Note that in the previous proof we proved that for every $X \subseteq S^Y$, X is a stationary set or $S^Y \backslash X$ is a stationary set.

Fact 3.12. *If $X, Y \neq \emptyset$ such that $X \subset Y$. Then for every $a \subset S^X$, $a \in U_X$ if and only if $\{Z \in Y : Z \cap X \in a\} \in U_Y$.*

Proof. $\Leftarrow$: Since $\{Z \in Y : Z \cap X \in a\} \in U_Y$, there exists $b \in \mathbb{Q}^S_{<\kappa}$ such that $b_Y = \{Z \in Y : Z \cap X \in a\}$ with $b \in G$, then

$$a = \{Z \cap X : Z \in b_Y\} = \{Z \cap X : \exists A \in b \wedge A \cap Y = Z\}$$

$$= \{A \cap Y \cap X : A \in b\} = \{A \cap X : A \in b\} = b_X \in U_X$$

$\Rightarrow$: Assume $\{Z \in Y : Z \cap X \in a\} \notin U_Y$, since U_Y is an ultrafilter on S^Y, $\{Z \in Y : Z \cap X \notin a\} \in U_Y$, therefore there exists $b \in \mathbb{Q}^S_{<\kappa}$, $b \in G$ such that $b_Y = \{Z \in Y : Z \cap X \notin a\} \in U_Y$, so $\{Z \cap Y : Z \in b\} = \{Z \in Y : Z \cap X \in a\}$.

Let $Z' \in b_X$ then $Z' = Z \cap X$ for some $Z \in b$, but $Z \cap Y \in b_Y$ then $Z \cap Y \cap X \notin a$. Therefore $Z \cap X \notin a$ and $b_X \subseteq S^X \backslash a$, since $b_X \in U_X$ and U_X is an ultrafilter, $a \notin U_X$. □

For each $\emptyset \neq X \in V_\kappa$ let $(M_X, E_X) = Ult(V, U_X)$ and let $j_X : V \to (M_X, E_X)$ be the induced embedding.
For $X \subset Y$ we define $j_{X,Y} : M_X \to M_Y$ as $j_{X,Y}([f]_{U_X}) = [f_Y]_{U_Y}$ where $f_Y(Z) = f(Z \cap X)$.
Let (M, E) be the limit of the family

$$< M_X, j_X, j_{X,Y} : X, Y \in V_\kappa \backslash \{\emptyset\}, X \subset Y >$$

and j the corresponding limit of the embeddings.
The following lemma describes the function that represents X in the generic ultra power when $X \in V_\kappa$.

Lemma 3.13. *For any $a \in \mathbb{Q}^S_{<\kappa}$ the identity function on a represents $j[\cup a]$ in the generic ultrapower, i.e. $\{b \in M : bE[i^{\cup a}_{\cup a}]_G\} = j[\cup a]$.*

Proof. Fix $a \in \mathbb{Q}^S_{<\kappa}$ and $b \in \mathbb{Q}^S_{<\kappa}$ such that $\cup a \subset \cup b$.
Let $i^{\cup a}_{\cup b} : S^{\cup b} \to S^{\cup a}$, $i^{\cup a}_{\cup b}(Z) = Z \cap (\cup a)$.

Claim: $[i^{\cup a}_{\cup b}]_{U_{\cup b}} = j_{\cup b}[\cup a]$.
Proof of the claim: Working on V.
$\supseteq$: Let $x \in \cup a$ and $f : S^{\cup b} \to \cup a$, $f(Z) = x$. To show $j_{\cup b}(x) \in [i^{\cup a}_{\cup b}]_{U_{\cup b}}$ it is enough to show $\{Z \subseteq \cup b : x \in Z\} \in U_{\cup b}$; by projection the elements of $U_{\cup b}$ are stationary sets in $\mathcal{P}(\cup b)$ and since $U_{\cup b}$ is an ultrafilter on $S^{\cup b}$, so it is enough to show that $S^{\cup b} \backslash \{Z \subseteq \cup b : x \in Z\}$ is not stationary. Assume it is stationary so $\cup(S^{\cup b} \backslash \{Z \subseteq \cup b : x \in Z\}) = \cup b$ then there exists $\mathcal{Z} \in S^{\cup b} \backslash \{Z \subseteq \cup b : x \in Z\}$ such that $x \in \mathcal{Z}$ a contradiction.
$\subseteq$: Let $[f]_{U_{\cup b}} \in [i^{\cup a}_{\cup b}]_{U_{\cup b}}$ in $M_{\cup b}$. So $f : S^{\cup b} \to \cup a$ is such that $f(Z) \in Z \cap (\cup a)$ if and only if $Z \in d_{\cup b}$ where $d \in G$ and $\cup b \subseteq \cup d$.

Subclaim: $\{p \in \mathbb{Q}^S_{<\kappa} : p \Vdash [f]_{U_{\cup b}} \in j_{\cup b}[\cup a]\}$ is dense in $\mathbb{Q}^S_{<\kappa}$.
Proof of the subclaim: Assume $c \in \mathbb{Q}^S_{<\kappa}$ and $(\cup d) \cup (\cup c) \subseteq Y$. By lifting in $\mathbb{Q}^S_{<\kappa}$, $c_1 = c^Y \leq c$, define $f^* : c_1 \to \cup a$, $f^*(Z) = f(Z \cap (\cup b))$ by normality in $\mathbb{Q}^S_{<\kappa}$, there exists $x \in \cup a$ such that $c' = \{Z \in c : f^*(Z) = x\}$, $c' \leq c$ in $\mathbb{Q}^S_{<\kappa}$, we conclude $c' \Vdash [f]_{U_{\cup b}} = j_{\cup b}(x)$, as we wanted.

By the subclaim and since G is generic, we conclude $[i^{\cup a}_{\cup b}]_{U_{\cup b}} \subseteq j_{\cup b}[\cup a]$, finishing the proof of the claim.

By the definition of j_Y we obtain $[i^X_{Y_0}]_{U_{Y_1}} = j_{Y_0,Y_1}([i^X_{Y_0}]_{U_{Y_0}})$ for $X \subseteq Y_0 \subset Y_1$. Using the claim we get $j_{Y_1}[\cup a] = j_{Y_0,Y_1}([i^{\cup a}_{Y_0}]_{U_{Y_0}})$ and in the limit for $Y_0 = \cup a$ we get $j[\cup a] = j^{\infty}_{\cup a}([i^{\cup a}_{\cup a}]_{U_{\cup a}}) = [i^{\cup a}_{\cup a}]_G$. □

Corollary 3.14. *For every $X \in V_\kappa$:*

1. $U_X = \{A \subseteq S^X : j[X] E j(A)\}$.

2. *For every $a \in \mathbb{Q}^S_{<\kappa}$, $a \in G$ if and only if $j[\cup a] E j(a)$.*

Proof. 1. By lemma 3.13, $\{A \subseteq S^X : j[X] E j(A)\} = \{A \subseteq S^X : [i^X_X]_G E j(A)\}$ and in the ultrapower this is $[i^X_X]_{U_X} E_X j_X(A)$, by Łos theorem this is $\{Z \in S^X : Z \in A\} \in U_X$. In the limit we have $A \in U_X$ if and only if $j[X] E j(A)$.

2. If $j[\cup a] \in j(a)$ then $a \in U_{\cup a}$, so there exists $b \in G$ so that $\cup a \subset \cup b$ and $b_{\cup a} = a$, then $a \geq b$ by the genericity of G, $a \in G$.
If $a \in G$, then $a \in U_{\cup a}$ and $j[\cup a] \in j(a)$. □

One of the most interesting cases of the Lemma 3.13 is the case when a is an ordinal smaller than κ. For those cases the function that represents them in the generic ultrapower is an order type function as the following lemma describes. This lemma has the corollaries 3.16 and 3.17 that show some

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ordinals inequalities in the generic ultrapower. In particular the corollary 3.17 shows that κ has cardinality at most λ in M.

Lemma 3.15. *For each $\alpha < \kappa$, the function $g_\alpha : S^\alpha \to \alpha$,*
$g_\alpha(Z) = o.t.(Z)$ represents α in M.

Proof. Claim: $M_\alpha \models ([g_\alpha]_{U_\alpha}$ is the transitive collapse of $[id^\alpha]_{U_\alpha})$.
Proof of the claim:

$$\{y \in S^\alpha : g_\alpha(y) \text{ is the transitive collapse of } id^\alpha(y)\} =$$

$$\{y \in S^\alpha : g_\alpha(y) \text{ is the order type of } id^\alpha(y)\} = S^\alpha$$

and $S^\alpha \in U_\alpha$.

Since $M_\alpha \xrightarrow{j_{\alpha,\infty}} M$ is elementary, by the claim we have
$M \models ([g_\alpha]$ is the transitive collapse of $[id^\alpha])$.
Let $h : (\mathrm{ext}^V_{(M,E)}([id^\alpha]), E) \to (\mathrm{ext}^V_{(M,E)}([g_\alpha]), E)$ be the function
$h(b) = c$ if $M \models (f(b) = c)$ where

$$M \models (f : [id^\alpha] \to [g_\alpha] \text{ is the collapsing map})$$

since the collapsing map is an isomorphism, then h is an isomorphism and $(\mathrm{ext}^V_{(M,E)}([id^\alpha]), E) \cong (\mathrm{ext}^V_{(M,E)}([g_\alpha]), E)$ by lemma 3.13 we get

$$(\mathrm{ext}^V_{(M,E)}([g_\alpha]), E) \cong (j[\alpha], E) \cong (\alpha, \in)$$

By the definition of $wfp(M, E)$,

$$wfp(M, E) = \{x \in M : \{b \in M : bE\ tc(x)^{(M,E)}\} \text{ is well founded in } V\}$$

then $[g_\alpha] \in wfp(M, E)$, since $[g_\alpha]$ is transitive, $[g_\alpha]$ is an ordinal in (M, E) isomorphic to α; we identify $[g_\alpha]$ with α □

Corollary 3.16. *Given a cardinal λ such that $\omega < \lambda \leq \kappa$, $S = \mathcal{P}_\lambda(V_\kappa)$ and G a V-generic. Fix $\beta < \kappa$, $\beta \subseteq X$, $a \in \mathbb{Q}^S_{<\kappa}$ stationary in $\mathcal{P}(X)$ and function $f : \mathcal{P}_\lambda(X) \to Ord$:*

1. *If* $\{Z \in \mathcal{P}_\lambda(X) : Z \notin a \vee o.t(Z \cap \beta) \geq f(Z)\}$ *contains a club in* $\mathcal{P}(X)$, *then* $a \Vdash \check{\beta} \geq [f]_{U_X}$.

2. *If* $\{Z \in \mathcal{P}_\lambda(X) : Z \notin a \vee o.t(Z \cap \beta) \leq f(Z)\}$ *contains a club in* $\mathcal{P}(X)$, *then* $a \Vdash \check{\beta} \leq [f]_{U_X}$.

Proof. It is enough to prove 1.
Let G be a generic such that $a \in G$. Assume
$\{Z \in \mathcal{P}_\lambda(X) : Z \notin a \vee o.t(Z \cap \beta) \geq f(Z)\}$ contains a club and $\check{\beta} \ngeq [f]_{U_X}$. By lemma 3.15 and the definition of $j_{\beta,X}$, $\{Z \in \mathcal{P}_\lambda(X) : o.t(Z \cap \beta) \geq f(Z)\} \notin U_X$, since U_X is an ultrafilter on $\mathcal{P}_\lambda(X)$, $\{Z \in \mathcal{P}_\lambda(X) : o.t(Z \cap \beta) < f(Z)\} \in U_X$ but
$\{Z \in \mathcal{P}_\lambda(X) : Z \notin a \vee o.t(Z \cap \beta) \geq f(Z)\} \in U_X$ so $\{Z \in \mathcal{P}_\lambda(X) : Z \notin a\} \in U_X$, contradiction since $a \in U_X$. □

Corollary 3.17. *Given a regular uncountable cardinal* λ *such that* $\lambda \leq \kappa$, $S = \mathcal{P}_\lambda(V_\kappa)$ *and* G *a* V*-generic. Then* $j(\lambda) \geq \kappa$ *in the generic ultrapower.*

Proof. It is enough to prove that for every $\beta < \kappa$, $\{Z \in \mathcal{P}_\lambda(\beta) : o.t(Z \cap \beta) \leq \lambda\} \in U_\beta$ but this is $\mathcal{P}_\lambda(\beta) \in U_\beta$. □

Remark. For $\mathbb{P}_{<\kappa}$, since $\mathbb{P}_{<\kappa} = \mathbb{Q}^S_{<\kappa}$ when we apply the previous corollary we obtain $j(\kappa) \geq \kappa$. A way to obtain the critical point of j is using corollary 3.14. Let $\gamma < \kappa$ be a regular uncountable cardinal, $F : [\gamma]^{<\omega} \to \gamma$ and $\beta < \gamma$. Since γ is regular $H_F(\beta) \neq \gamma$, $H_F(\beta)$ is a limit ordinal, $\gamma \cap C_F \neq \emptyset$, γ is stationary in $\mathcal{P}(\gamma)$, $\gamma \in \mathbb{P}_{<\kappa}$. If $\gamma \in G$ then $j[\gamma] \in j(\gamma)$ and $j[\gamma]$ is an ordinal below $j(\gamma)$, then $j[\gamma]$ is transitive thus $j(\alpha) = \alpha$ for every $\alpha < \gamma$ and the critical point of j is γ.

Corollary 3.18. *Assume that* μ *and* λ *are regular uncountable cardinals such that* $\mu \leq \lambda \leq \kappa$, $S = \mathcal{P}_\lambda(V_\kappa)$ *and* G *a* V*-generic. Let* $a_\mu = \{z \in \mathcal{P}_\lambda(\mu) : z \cap \mu \in \mu\}$. *Then* $cp(j) = \mu$ *if and only if* $a_\mu \in G$.

Proof. By the lemma 3.13, $a \in G$ if and only if $j[\mu] \cap j(\mu) \in j(\mu)$. Therefore $j[\mu] \cap j(\mu)$ is an ordinal below $j(\mu)$. We conclude that $a \in G$ if and only if $cp(j) = \mu$. □

Note that for the towers $\mathbb{R}^{\lambda}_{\kappa}$, for every G V-generic we have $cp(j) = \lambda$.
Before finishing the chapter we will show the reason why we haven't worked with the towers $\mathbb{R}^{\lambda}_{\kappa}$ during the chapter.
When λ is a regular uncountable cardinal, and $\kappa < \kappa'$ strongly inaccessible cardinals as in $\mathbb{R}^{\lambda}_{\kappa}$. In the previous remark we saw that λ is stationary and $\lambda \in \mathbb{Q}^{\mathcal{P}_{\lambda}(V_{\kappa'})}_{<\kappa'}$. Since this tower is closed under lifting we get $(\mathcal{R}_{\lambda})^{V_{\kappa}} = \lambda^{V_{\kappa}} \in \mathbb{Q}^{\mathcal{P}_{\lambda}(V_{\kappa'})}_{<\kappa'}$ so $(\mathcal{R}_{\lambda})^{V_{\kappa}}$ is stationary and $\mathbb{R}^{\lambda}_{\kappa} = \mathbb{Q}^{\mathcal{R}_{\lambda}}_{<\kappa}$.

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Chapter 4

Large cardinals and stationary towers

" It's kind of fun to do the impossible.

- Walt Disney "

4.1 Completely Jónsson cardinals

This section is devoted to prove the corollary 4.13 that is one of the main properties of the stationary tower. The proof of the corollary 4.12 in [Lar04] (Lemma 2.3.2) doesn't use Ramsey cardinals, we are presenting a different proof using Ramsey cardinals.
If the reader is familiar with measurable cardinals and Ramsey cardinals, then the reader may skip the following two pages of basic acknowledge and continue with the the definition of Jónsson cardinal (definition 4.5).

κ is a Ramsey cardinal if for every partition F of $[\kappa]^{<\omega}$ into two pieces, $F : [\kappa]^{<\omega} \to 2$, there exists $H \subset \kappa$ such that $o.t.(H) = \kappa$ and for every $n \in \omega$, $|F[[H]^n]| = 1$. We denote this property by $\kappa \to (\kappa)^{<\omega}$.
κ an uncountable cardinal is measurable if there exists a κ-complete non-principal ultrafilter on κ. Given a non-trivial elementary embedding $j : V \to M$ with $cp(j) = \kappa$, the set $U \subset \mathcal{P}(\kappa)$, whose elements are the sets $X \subseteq \kappa$ that satisfy $\kappa \in j(X)$, is a κ-complete non-principal ultrafilter on κ.

For $< X_\alpha : \alpha < \kappa >$ a sequence of subsets of κ, we define the diagonal intersection as $\triangle_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$
A filter F on κ is normal if it is closed under diagonal intersection .
We call a normal measure on κ, a κ-complete non-principal ultrafilter on κ.

Lemma 4.1. *For every measurable κ there exists a normal measure on κ.*

Proof. Let U be a κ-complete non-principal ultrafilter on κ.
Since the relation $\sim$ in κ^κ, $f \sim g \iff \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$, has the property that $\kappa / \sim$ is well founded, we can define $f : \kappa \to \kappa$ a function such that $\{\alpha < \kappa : f(\alpha) > \gamma\} \in U$ for all γ, and $[f]$ is the least class with this property. Define

$$D = \{X \subseteq \kappa : f^{-1}(X) \in U\}$$

D is an ultrafilter, because $X \notin D$ implies $f^{-1}(X) \notin U$ so $f^{-1}(\kappa \backslash X) \in U$, $\kappa \backslash X \in D$.
D is κ-complete, because of the κ-completeness of U,
$f^{-1}(\bigcap_{\alpha<\kappa} X_\alpha) = \bigcap_{\alpha<\kappa} f^{-1}(X_\alpha) \in U$ so $\bigcap_{\alpha<\kappa} X_\alpha \in D$.
D is non-principal because for every $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$ so $f^{-1}(\{\gamma\}) \notin U$, $\{\gamma\} \notin D$.

Claim: For F a κ-complete non-principal ultrafilter on κ. If every regressive function $f : X \to \kappa$ for $X \in F$, is constant in a set $Y \in F$, then F is closed under diagonal intersection.
Proof of the claim: Assume there exists $< X_\alpha : \alpha < \kappa >$, $X_\alpha \in F$ such that $\triangle_{\alpha<\kappa} X_\alpha \notin F$. Then $\kappa \backslash \triangle_{\alpha<\kappa} X_\alpha \in F$, let f be the function $f : \kappa \backslash \triangle_{\alpha<\kappa} X_\alpha \to \kappa$, $f(\alpha) = \xi$ where $\xi < \alpha$ and $\alpha \notin X_\xi$. For $\alpha \in \kappa \backslash \triangle_{\alpha<\kappa} X_\alpha$ we have $\alpha \notin \bigcap_{\beta<\kappa} X_\beta$ so there exists $\xi < \alpha$ such that $\alpha \notin X_\xi$, and f is well defined. By the assumptions of the claim, there exists $\gamma < \kappa$ and $Y \in F$ such that $f[Y] = \{\gamma\}$, then $Y \cap X_\gamma = \emptyset \in F$, contradiction.

To prove that D is normal, it is enough to prove that for every regressive

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function $g : X \to \kappa$ on $X \in D$ there exists $Y \in D$ such that g is constant on Y.

Let $h : f^{-1}(X) \to \kappa$, $h(\alpha) = g(f(\alpha))$. Since g is regressive and $f^{-1}(X) \in U$ we have $[h] < [f]$, let $\gamma < \kappa$ be the least ordinal such that $\{\alpha : h(\alpha) \leq \gamma\} \in U$ by the minimality of f such γ exists, let $X_\alpha = \{\beta : h(\beta) > \alpha\}$ for $\alpha < \gamma$ and $X_\gamma = \{\beta : h(\beta) \leq \gamma\}$ by the κ-completeness of U,

$\bigcap\limits_{\alpha \leq \gamma} X_\alpha = \{\beta : h(\beta) = \gamma\} \in U$ so g is constant on $f(\bigcap\limits_{\alpha \leq \gamma} X_\alpha)$. □

Theorem 4.2. *Every measurable cardinal is a Ramsey cardinal.*

Proof. Let D be a normal measure on κ.

By induction we will show that for any partition $F : [\kappa]^{<\omega} \to 2$ there exists $H_n \in D$ such that $|F[[H_n]^n]| = 1$. For $n = 1$ it is clear. Assume it is true for n. Let $F : [\kappa]^{<\omega} \to 2$ a partition. Define $F_\alpha : [\kappa \setminus \{\alpha\}]^n \to 2$ as $F_\alpha(X) = F(\{\alpha\} \cup X)$ by the induction hypothesis there exists $X_\alpha \in D$ such that F_α is constant on X_α.

For any $\beta_1 < \beta_2 < \cdots < \beta_{n+1} \in \triangle_{\alpha<\kappa} X_\alpha$ we have

$F(\{\beta_1, \ldots, \beta_{n+1}\}) =$

$F_{\beta_1}(\{\beta_2, \ldots, \beta_{n+1}\}) = i_{\beta_1}$, only depends on β_1. We conclude that $\triangle_{\alpha<\kappa} X_\alpha = Y_0 \cup Y_1$, such that $\forall \alpha, \beta \in Y_j$, $i_\alpha = i_\beta$ for $j \in \{0, 1\}$. Since D is normal, $\triangle_{\alpha<\kappa} X_\alpha \in D$ then $Y_0 \in D$ or $Y_1 \in D$, let H_{n+1} be the one in D, then F is constant in $[H_{n+1}]^{n+1}$.

To finish the proof, given a partition $F : [\kappa]^{<\omega} \to 2$ let H_n be the set such that $f \restriction_{[H_n]^n}$ is constant. Then for every $n \in \omega$ f, is constant on $[\bigcap\limits_{i=1}^{\infty} H_i]^n$. □

Lemma 4.3. *If κ is a measurable cardinal, U a κ-complete ultrafilter on κ and $j : V \to Ult(V, U)$ the induced embedding, then $cp(j) = \kappa$.*

Proof. Note that $f_\alpha \kappa \to \{\alpha\}$ for $\alpha \leq \kappa$ represents α in $Ult(V, U)$, let $i : \kappa \to \kappa$ be the identity, so $[f_\alpha]_U < [i]_U < [f_\kappa]_U$ for $\alpha < \kappa$, $j(\kappa) > \kappa$ and using Łos theorem and assuming $[f_\alpha] \neq j(\alpha)$, there exists $[g] \in [f_\alpha]$ such that the sets

$\{x \in \kappa : g(x) \in \alpha\}$ and $\{x \in \kappa : g(x) \notin \beta\}$ for $\beta < \alpha$, are in the ultrafilter, but is κ-complete, so $\emptyset \in U$ contradiction. $\square$

Theorem 4.4. *Every measurable cardinal is a Ramsey cardinal limit of Ramsy cardinals.*

Proof. Let U be a κ-complete ultrafilter on κ, $M = Ult(V, U)$, $j : V \to M$ the induced embedding and $cp(j) = \kappa$. Since $M \subseteq V$, $\mathcal{P}(\kappa)^M \subseteq \mathcal{P}(\kappa)^V$. Let $X \in \mathcal{P}(X)^V$, if $|X| < \kappa$ since $cp(j) = \kappa$ then $X \in \mathcal{P}(\alpha)^M$, for some $\alpha < \kappa$, $X \in \mathbb{P}(\kappa)^M$; for $|X| = \kappa$, note that $j(X) \cap \kappa = X$, thus X can be calculated on M, so $X \in M$, we conclude $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^V$.

Claim: $V^M_{\kappa+1} = V_{\kappa+1}$.
Proof of the claim: Let $X \in V_{\kappa+1}$, so $X \subseteq V_\kappa$, let f be such that

$$M \models (f : \kappa \to V_\kappa \text{ is a bijection })$$

and $Y = f^{-1}[X]$, since $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^V$ we get $Y \in M$ and $f(Y) = X \in M$.

Let $F : [\kappa]^{<\omega} \to 2$ in M, but $f \in V^M_{\kappa+1} = V_{\kappa+1}$ since $V \models (\kappa \text{ is Ramsey})$, there exists $H \in [\kappa]^\kappa$ such that f is constant in $[H]^n$ for every $n \in \omega$. But $H \in \mathcal{P}(\kappa)^V = \mathcal{P}(\kappa)^M$, we conclude $M \models (\kappa \text{ is Ramsey})$, so
$M \models (\forall \alpha < \kappa\ \exists \beta(\alpha < \beta < j(\kappa) \wedge \beta \text{ is Ramsey}))$. We conclude κ is a limit of Ramsey cardinals. $\square$

The completely Jónsson cardinals give the name to this section because they are the ones that ensure the corollary 4.13. The reason why we introduce the Ramsey cardinals and the measurable cardinals in the previous part is because these cardinals ensure the existence of a completely Jónsson cardinal that is a limit of completely Jónsson cardinals. Before showing this, let's show (as the reader can expect) that every completely Jónsson cardinal is a Jónsson cardinal.

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An algebra is a structure $< A, f_n >_{n\in\omega}$ where $f_n : [A]^{g(n)} \to A$ for some $g(n) \in \omega$, and a subalgebra is a structure $< A_0, f_n \restriction_{[A_0]^{g(n)}} >_n \in \omega$ where $A_0 \subset A$ and $f_n[[A_0]^{g(n)}] \subseteq A_0$.

Definition 4.5. *κ is a Jónsson cardinal if every algebra of cardinality κ has a proper subalgebra of cardinality κ.*

Definition 4.6. *A strongly inaccessible cardinal κ is a completely Jónsson cardinal if for every $a \in \mathbb{P}_{<\kappa}$ the set*

$$\{X \subset V_\kappa : X \cap (\cup a) \in a \wedge |X \cap \kappa| = \kappa\}$$

is stationary in $\mathcal{P}(V_\kappa)$.

Fact 4.7. *If for every $a \in \mathbb{P}_{<\kappa}$*

$$A_a = \{X \subseteq V_\kappa : X \cap (\cup a) \in a \wedge |X \cap \kappa| = \kappa\}$$

is a stationary set in $\mathcal{P}(V_\kappa)$, then $A' = \{X \subsetneq V_\kappa | X \cap \kappa| = \kappa\}$ is a stationary set in $\mathcal{P}(V_\kappa)$.

Proof. $A_a \subseteq A'$ if and only if $V_\kappa \notin A_a$, let a be a regular uncountable cardinal, by a previous remark $a \in \mathbb{P}_{<\kappa}$ and $V_\kappa \notin A_a$. So A' is a stationary set in $\mathcal{P}(V_\kappa)$. □

Lemma 4.8. *Suppose $A \subseteq V_\kappa$ such that $|A| = \kappa$, let*

$$Y_A = \{X \subsetneq V_\kappa : |X \cap A| = \kappa\}$$

then Y_A is stationary if and only if Y_κ is stationary.

Proof. Let $h : V_\kappa \to V_\kappa$ be a bijection with $h(\kappa) = A$. Let $F : [V_\kappa]^{<\omega} \to V_\kappa$, $F' = h^{-1} \circ F \circ h$. So $X \in Y_\kappa$ if and only if $|X \cap \kappa| = \kappa$, that happens if and only if $|X \cap h^{-1}[A]| = \kappa$, that happens if and only if $|h[X] \cap A| = \kappa$, that happens if and only if $h[X] \in Y_A$. For every $X \in C_{F'}$ and $x_1, x_2, \ldots, x_n \in X$ we get $F(\{h(x_1), h(x_2), \ldots, h(x_n)\}) = h(F'(\{x_1, x_2, \ldots, x_n\})) \in h[X]$, $h[X] \in C_F$. In the same way $h[X] \in C_F$ implies $X \in C_{F'}$. We conclude $X \in Y_\kappa \cap C_{F'}$ if and only if $h[X] \in Y_A \cap C_F$. □

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Theorem 4.9. *Every completely Jónsson cardinal is a Jónsson cardinal.*

Proof. Let $< A, f_n >_{n\in\omega}$ be an algebra with $|A| = \kappa$, we can assume $A = V_\kappa$. Let C_n be the club associated to F_n, where $F_n : [V_\kappa]^{<\omega} \to V_\kappa$, $F_n(Z) = f_n(Z)$ if $Z \in [V_\kappa]^{g(n)}$ and $F_n(Z) = \emptyset$ in other case.
Let C be the club $C = \bigcap_{n\in\omega} C_n$. Since κ is a completely Jónsson, by the fact 4.7 and the lemma 4.8 we have that $\{X \subsetneq V_\kappa : |X| = \kappa\}$ is a stationary set in $\mathcal{P}(V_\kappa)$, so $C \cap \{X \subsetneq V_\kappa : |X| = \kappa\} \neq \emptyset$, let $Y \in C \cap \{X \subsetneq V_\kappa : |X| = \kappa\}$ then $< Y, f_n >_{n\in\omega}$ is a proper subalgebra of cardinality κ. □

Fact 4.10. *κ is a Ramsey cardinal if and only if for every $\gamma < \kappa$, $\kappa \to (\kappa)^{<\omega}_\gamma$, i.e. for every partition $f : [\kappa]^{<\omega} \to \gamma$, there is $H \in [\kappa]^\kappa$ such that for every $n \in \omega$ $|f[[H]^n]| = 1$.*

Proof. Let $f : [\kappa]^{<\omega} \to \gamma$, $\gamma < \kappa$, be a partition.
Define $g : [\kappa]^{<\omega} \to 2$, $g(\xi_1, \xi_2, \ldots, \xi_n) = 0$ if $n = 2m$, and 1 in other case. Let $H \in [\kappa]^\kappa$, $|g[[H]^n]| = 1$ for any $n \in \omega$. Fix $n \in \omega$ since $\gamma < \kappa$, there exist $a, b \in [H]^n$ such that $f(a) = f(b)$ and $\max(a) < \min(b)$, so $g(a \cup b) = 0$, then $g[[H]^{2n}] = 0$. For $x, y \in [H]^n$ there exists $z \in [H]^n$ such that
$max(x \cup y) < \min(z)$, $g(x \cup z) = g(y \cup z) = 0$, then $f(x) = f(y)$, $|f[[H]^n]| = 1$ □

Lemma 4.11. *Every Ramsey cardinal is a completely Jónsson cardinal.*

Proof. Fix $a \in \mathbb{P}_\kappa$ and $H : [V_\kappa]^{<\omega} \to V_\kappa$. Let $\mathcal{L}$ be the language with a predicate for H and a constant for each element of $\cup a$. Since $a \in V_\kappa$, $|\mathcal{L}| = \alpha < \kappa$ and since κ is inaccessible, the set T of types in $\mathcal{L}$ has cardinality $\gamma < \kappa$. Let $g : T \to \gamma$ be a bijection, define $F : [\kappa]^{<\omega} \to \gamma$, $F(a) = g(tp(\bar{a}))$. By the fact 4.10, there exists $I \subsetneq [\kappa]^\kappa$ such that F is constant on $[I]^n$ for every $n \in \omega$, so I is a set of indiscernibles ordinals.
Let $f : \omega \to \omega \times \omega$ be a surjection such that $f(k)_1 \leq k$ for all k.
Let $(H_n)_{n\in\omega}$ be an enumeration of the terms obtained by iterated applications of H, as in the proof of lemma 2.11. Fix $a_0 \in \cup a$ and define

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$H^* : [\cup a]^{<\omega} \to \cup a$ as $H^*(x_0, \ldots, x_k) = h_{f(k)_0}(x_0, \ldots, x_{f(k)_1-1}, i_0, \ldots, i_t)$ if this belongs to $\cup a$ and a_0 in other case.

Let $X' \in C_{H^*} \cap a$, and $X = cl_H(X' \cup I)$. Clearly $X \in C_H$ and $|X| = \kappa$. Suppose $x \in X \cap (\cup a)$, then there exist $x_0, x_1, \ldots, x_n \in X'$ and $m \in \omega$, such that $x = H_m(x_0, \ldots, x_n, i_0, \ldots, i_t)$, since I is a set of indiscernibles, x doesn't depend on $i_0, \ldots, i_t$. There exists $k \in \omega$ such that $f(k) = (m, n+1)$ and by definition of H^* we have $H^*(x_0, \ldots, x_n, x_0, \ldots x_0) = x$, where x_0 appears $k - f(k)_1 + 1$ times, we conclude $x \in X'$. We have shown $X \cap (\cup a) = X' \in a$, we conclude that κ is a completely Jónsson cardinal. □

Corollary 4.12. *If κ is a measurable cardinal then κ is a completely Jónsson cardinal limit of completely Jónsson cardinals.*

Proof. Follows from theorem 4.4 and lemma 4.11. □

Corollary 4.13. *Given a cardinal λ such that $\omega < \lambda \leq \kappa$ and $S = \mathcal{P}_\lambda(V_\kappa)$, suppose that κ is a limit of completely Jónsson cardinals. Let j be a generic embedding corresponding to $\mathbb{Q}^S_{<\kappa}$. Then $j(\lambda) = \kappa$.*

Proof. Let $\alpha < j(\lambda)$.

Claim: The set $\{b \in \mathbb{Q}^S_{<\kappa} : b \Vdash \check{\alpha} < \check{\kappa}\}$ is dense in $\mathbb{Q}^S_{<\kappa}$.

Proof of the claim: Let $a \in \mathbb{Q}^S_{<\kappa}$. Take $X \in V_\kappa$ such that there exists a function $f : S^X \to \lambda$ that represents α in U_X, let $a' = a^X$, clearly $a' \leq a$, and let $\gamma < \kappa$ be a completely Jónsson such that $a' \in V_\gamma$.

Define b as the set of sets $|X| < \lambda$, $X \subset V_\gamma$ that satisfies:

- $X \cap (\cup a) \in a$.
- $f(X \cap (\cup a)) \leq o.t.(X \cap \gamma)$.

Fix a function $F : [V_\gamma]^{<\omega} \to V_\gamma$, since γ is completely Jónsson, $\{X \subset V_\gamma \cap (\cup a) \in a \wedge |X \cap \gamma| = \gamma\} \cap C_F \neq \emptyset$; let Y be an element of that set and Z a subset of $Y \cap \gamma$ such that $|Z| = \lambda$ and $o.t.(Z) \geq f(Y \cap (\cup a))$.

Since $Y \in C_F$, we get $Y \cap (\cup a) \subseteq cl_F((Y \cap (\cup a)) \cup z) \cap (\cup a) \subseteq Y \cap (\cup a)$, then $cl_F((Y \cap (\cup a)) \cup z) \in b$. Therefore $b \in \mathbb{P}_{<\kappa}$ and since all its elements has cardinality less than λ, $b \subseteq \mathcal{P}_{\omega_1}(V_\kappa) \subseteq S$, we conclude $b \in \mathbb{Q}^S_{<\kappa}$, $b \leq a'$ and $b \Vdash [f]_{U_X} \leq \gamma$.

By the claim we get $j(\lambda) \leq \kappa$. The other inequality is the corollary 3.17. $\square$

4.2 Properties of $\mathbb{Q}^S_{<\kappa}$ when $S = \mathcal{P}_\lambda(V_\kappa)$

From now on we are going to focus on the case when S is of the form $\mathcal{P}_\lambda(V_\kappa)$, for some regular uncountable cardinal $\lambda \leq \kappa$.

The first property that we are going to study is when we have to cardinals $\kappa_1 < \kappa_2$ what happens to the generic filters of $\mathbb{Q}^{S_2}_{<\kappa_2}$ when we restrict them to the tower $\mathbb{Q}^{(S_2)_{V_{\kappa_1}}}_{<\kappa_1}$.

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Lemma 4.14. *Let $\lambda \leq \kappa_1$. Suppose that $\kappa_1 < \kappa_2$ are strongly inaccessible cardinals, $S_1 = \mathcal{P}_\lambda(V_{\kappa_1})$ and $S_2 = \mathcal{P}_\lambda(V_{\kappa_2})$.*
Let $G \subset \mathbb{Q}^{S_2}_{<\kappa_2}$ be a V-generic such that $G \cap \mathbb{Q}^{S_1}_{<\kappa_1}$ is also a V-generic, and let a be the set of sets $|X| < \lambda$, $X \prec V_{\kappa_1+1}$ such that for every predense $D \subset \mathbb{Q}^{S_1}_{<\kappa_1}$ if $D \in X$ then $X \cap (\cup d) \in d$ for some $d \in X \cap D$.
Then $a \in G$.

Proof. Let $j : V \to (M, E)$ be the generic embedding resulting from $\mathbb{Q}^{S_2}_{<\kappa_2}$ with the generic G.
Suppose $Z \in j[V_{\kappa_1+1}] = j[\cup a]$ is predense in $j(\mathbb{Q}^{S_1}_{<\kappa_1})$, so $Z = j(D)$ for some predense D in $\mathbb{Q}^{S_1}_{<\kappa_1}$. By assumption $G \cap \mathbb{Q}^{S_1}_{<\kappa_1}$ is V-generic so $G \cap D \neq \emptyset$, let $d \in G \cap D$ so $j(d) \in j[V_{\kappa_1+1}]$.
Since $(\cup j(d)) \cap j[V_{\kappa_1+1}] = j[\cup d]$ and $d \in G$, by the corollary 3.14 (2), $(\cup j(d)) \cap j[V_{\kappa_1+1}] \in j(d)$. By corollary 3.17, $|j[V_{\kappa_1+1}]| < \kappa_2 \leq j(\lambda)$, we conclude that in M, $|j[V_{\kappa_1+1}]| < \lambda$, $j[V_{\kappa_1+1}] \prec V_{\kappa_1+1}$ and for every Z predense

in $j(\mathbb{Q}^{S_1}_{<\kappa_1})$, if $Z \in j[V_{\kappa_1+1}]$ then there exists $j(d) \in Z \cap j[V_{\kappa_1+1}]$ such that $(\cup j(d)) \cap j[V_{\kappa_1+1}] \in j(d)$, so $j[V_{\kappa_1+1}] \in j(a)$ and by corollary 3.14, $a \in G$. $\square$

Lemma 4.15. *Let $\lambda \leq \kappa_1$. Suppose that $\kappa_1 < \kappa_2$ are strongly inaccessible cardinals, $S_1 = \mathcal{P}_\lambda(V_{\kappa_1})$ and $S_2 = \mathcal{P}_\lambda(V_{\kappa_2})$.*
Let a be the set of sets $|X| < \lambda$, $X \prec V_{\kappa_1+1}$ such that for every predense $D \subset \mathbb{Q}^{S_1}_{<\kappa_1}$ if $D \in X$ then $X \cap (\cup d) \in d$ for some $d \in X \cap D$. If a is stationary, then a forces that $G \cap \mathbb{Q}^{S_1}_{<\kappa_1}$ will be V-generic for $\mathbb{Q}^{S_1}_{<\kappa_1}$, where G is the generic filter for $\mathbb{Q}^{S_2}_{<\kappa_2}$

Proof. Let $G \in \mathbb{Q}^{S_2}_{<\kappa_2}$ be a V-generic with $a \in G$. It is easy to check that $G \cap \mathbb{Q}^{S_1}_{<\kappa_1}$ is a filter using lifting and projection.
Let $D \subset \mathbb{Q}^{S_1}_{<\kappa_1}$ be a maximal antichain in $\mathbb{Q}^{S_1}_{<\kappa_1}$ and assume that $D \cap G = \emptyset$. Since $\mathbb{P}^{S_2}_{<\kappa_2}$ is dense and $H = \{X \in S_2 : D \in X\}$ is a club, the set $\{X \cap H : X \in \mathbb{P}^{S_2}_{<\kappa_2}\}$ is dense in $\mathbb{Q}^{S_2}_{<\kappa_2}$ so $\{X \cap H : X \in \mathbb{P}^{S_2}_{<\kappa_2}\} \cap G \neq \emptyset$, let b be an element of this set, then there exists $c \in G$ such that $c \leq a$, $c \leq b$ this means that for every $Z \in c$, $Z \cap (\cup b) \in b$ so $D \in Z$ and $c \cap (\cup a) \in a$, therefore $c_{\cup a} \leq a_D = \{X \in a : D \in X\}$, and $a_D \in G$.

Claim: D is an antichain in $\mathbb{Q}^{S_2}_{<\kappa_2}$.
Proof of the claim: Assume that there exist $d_1, d_2 \in D$ compatibles in $\mathbb{Q}^{S_2}_{<\kappa_2}$, so there exists $c \in \mathbb{Q}^{S_2}_{<\kappa_2}$ such that $c \leq d_1$, $c \leq d_2$ then $c_{(\cup d_1)\cup(\cup d_2)} \leq d_1$ and $c_{(\cup d_1)\cup(\cup d_2)} \leq d_2$ but $c_{(\cup d_1)\cup(\cup d_2)} \in \mathbb{Q}^{S_1}_{<\kappa_1}$ a contradiction.

Let A be a maximal antichain in $\mathbb{Q}^{S_2}_{<\kappa_2}$ such that $D \subseteq A$, since G is generic, there exists $b \in A \cap G$, but $a_D \in G$ so there exists $c \in G$ such that $c \leq a_D$ and $c \leq b$, so for every $Y \in c$, $Y \cap (\cup a_D) \in a_D$ then $D \in Y$ and there exists $d \in D \cap Y \cap (\cup a_D)$ such that $Y \cap (\cup a_D) \cap (\cup d) \in d$. Since $d \in D$ implies $\cup d \subset \cup a_D = \cup a$, we can define the following function.
Define $F : c \to D$, $F(Y) = d$ with $Y \cap (\cup d) \in d$. By normality there exists $d \in D$ such that $c' = \{Y \in c : F(Y) = d\} \in \mathbb{Q}^{S_2}_{<\kappa_2}$ and $c' \leq c$. Since $d \in V_{\kappa_1}$ we get $\cup d \subset V_{\kappa_1+1} = \cup a_D \subset \cup c = \cup c'$ so $c' \leq d$, but $c' \subseteq c \subseteq b^{\cup c}$ so $c' \leq b$,

contradicting that $d, b \in A$, a maximal antichain. □

With these two lemmas we can conclude that $G \cap \mathbb{Q}^{S_1}_{<\kappa_1}$ is a generic filter if and only if $a \in G$. Thus for future applications we will only check this element if it is in G.

The semi-proper sets have some interesting properties that are really useful when we want to work with some stationary towers and apply its properties, with these sets it is possible to apply the previous lemmas easily. The next theorems are the principal application of these sets, but these are not the only way to use the semi-proper sets, the proof of the theorem 5.9 is based on the idea of a semi-proper set.

Definition 4.16. *For a predense subset $D \subseteq \mathbb{Q}^{S}_{<\kappa}$, we define $sp(D)$ as the set of sets $X \prec V_{\kappa+1}$, $|X| < \lambda$ such that:*

- *There exists $Y \prec V_{\kappa+1}$ such that $X \subseteq Y$ and $|Y| < \lambda$.*
- *$Y \cap V_\kappa$ end-extends $X \cap V_\kappa$.*
- *$Y \cap (\cup d) \in d$ for some $d \in Y \cap D$*

D is semi-proper if $sp(D)$ contains a club of $\mathcal{P}_\lambda(V_{\kappa+1})$.

One interesting property of these sets is that one can find for every stationary set $d \in \mathbb{Q}^{S_1}_{<\kappa_1}$ a condition stronger that d in $\mathbb{Q}^{S_2}_{<\kappa_2}$.
If we assume that $D \subseteq \mathbb{Q}^{S}_{<\kappa}$ is a predense subset that is semi-proper and $d \in D$, then the set $X_d = \{X \in sp(D) : d \in X \wedge X \cap (\cup d) \in d\}$ is stationary; to see this assume that there exists $g : V^{<\omega}_{\kappa+1} \to V_{\kappa+1}$ such that $C_g \cap X_d = \emptyset$; since D is semi-proper and we know that there exists $H : V^{<\omega}_{\kappa+1} \to V_{\kappa+1}$ such that every $X \in C_H$ satisfies $X \in C_g$, $X \in sp(D)$ and $d \in X$; by the proof of the theorem 2.11, $(C_H)_{\cup d}$ contains a club C of $\cup d$, since d is stationary, $d \cap C \neq \emptyset$, therefore there exists $X \in (C_H)_{\cup d}$ such that $X = Y \cap (\cup d) \in d$ for some $Y \in C_H$ then $d \in Y$ and $Y \in X_d$: contradiction, since $C_H \cap X_d = \emptyset$.

Definition 4.17. *A cardinal δ is a Woodin cardinal if for all $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\theta < \delta$ there exists an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \theta$, $V_\theta \subset M$, and $A \cap V_\theta = j(A) \cap V_\theta$.*

It follows that if δ is a Woodin cardinal then there exists an unbounded set of measurable cardinals, and by corollary 4.12, δ is a limit of completely Jónsson cardinal. Thus $j(\lambda) = \delta$.

The following lemma is a well known result, the result is an if and only if but we are going to show only the direction that is useful for us.

Lemma 4.18. *If δ is a Woodin cardinal, then for each function $f : \delta \to \delta$ there exists an elementary embedding $j : V \to M$ with critical point $\gamma < \delta$ such that $f[\gamma] \subset \gamma$ and $V_{j(f)(\gamma)} \subset M$.*

Proof. Given a function $f : \delta \to \delta$, let $A = f$, take γ a cardinal that testifies that δ is a Woodin cardinal for A and let $\alpha = sup\{f(\rho) + 1 : \rho \leq \gamma\}$. Since $j(f) \cap V_\alpha = f \cap V_\alpha$ and $f(\gamma) < \alpha$ then $j(f)(\gamma) \in V_\alpha$ and since $V_\alpha \subset M$ then $V_{j(f)(\gamma)} \subset M$; suppose exists $\rho < \gamma$ such that $\gamma \leq f(\rho) < \alpha$ since $cp(j) = \gamma$ we get $\alpha < j(\gamma) < j(f)(\rho)$, a contradiction, f is closed in γ. □

The Woodin cardinal lets us use some nice properties of the stationary tower, one of this properties is that the towers $\mathbb{Q}^S_{<\delta}$ are well founded (theorem 4.22). The following results can be obtain with other cardinals that are not Woodin, see [Lar04], [Fuc10].

Theorem 4.19. *Suppose that δ is a Woodin cardinal. For each sequence $< D_\alpha : \alpha < \delta >$ of predense subsets of $\mathbb{Q}^S_{<\delta}$ there exists a strongly inaccessible cardinal $\gamma < \delta$ such that for every $\alpha < \gamma$, $D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is predense in $\mathbb{Q}^S_{<\gamma}$ and semi-proper.*

Proof. Let $f : \delta \to \delta$ be a increasing function such that

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1. If $\gamma < \delta$ is an ordinal closed under f, then for $\alpha < \delta$, $D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is predense in $\mathbb{Q}^S_{<\gamma}$.

2. For $\gamma < \delta$ an ordinal closed under f. If $\alpha < \delta$ is such that $D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is not semi-proper then exists $b \in D_\alpha \cap V_{f(\gamma)}$ compatible with

$$a = \{X \prec V_{\gamma+1} : |X| < min(\gamma, \lambda) \wedge X \notin sp(D_\alpha \cap \mathbb{Q}^S_{<\gamma})\}$$

$a \in \mathbb{Q}^S_{<\gamma}$ since $D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is not semi-proper, $a \subset \mathcal{P}_\lambda(V_{\gamma+1})$ and $\{X : X \prec V_{\gamma+1}\}$ is a club.

By the lemma 4.18, exists $j : V \to M$ with $cp(j) = \gamma < \delta$, $f[\gamma] \subset \gamma$ and $V_{j(f)(\gamma)+\omega} \subset M$. Therefore by (1) $D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is predense in $\mathbb{Q}^S_{<\gamma}$ for $\alpha < \gamma$, $j(D_\alpha) \cap \mathbb{Q}^S_{<\gamma} = D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ and $V_{\gamma+\omega} \subset M$, so $\mathcal{P}_\gamma(V_{\gamma+1}) \backslash sp(D_\alpha \cap \mathbb{Q}^S_{<\gamma})$ is stationary in $\mathcal{P}_\gamma(V_{\gamma+1})$ if and only if $M \models \mathcal{P}_\gamma(V_{\gamma+1}) \backslash sp(D_\alpha \cap \mathbb{Q}^S_{<\gamma})$ is stationary in $\mathcal{P}_\gamma(V_{\gamma+1})$.

Let a be as in (2) and assume $D_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is not semi-proper, then exists $b \in j(D_\alpha) \cap M_{j(f)(\gamma)}$ compatible with a in $j(\mathbb{Q}^S_{<\gamma})$. Since $V_{j(f)(\gamma)+\omega} \subset M$ then b is stationary in V and $c = \{X \subset (\cup a) \cup (\cup b) : X \cap (\cup a) \in a \wedge X \cap (\cup b) \in b\}$ is stationary in $\mathcal{P}((\cup a) \cup (\cup b))$.

Choose $j(\delta) < \eta$ a regular cardinal such that $V_\delta, D_\alpha \in V_\eta$. Since the sets $X \prec V_\eta$ form a club, the sets X such that $\{a, b, j \upharpoonright V_{\gamma+1}, j(V_{\gamma+2})\} \in X$ is a club and c is a stationary set, exists $X \in c$ such that $X \cap (\cup c) \in c$, $X \prec V_\eta$ and $\{a, b, j \upharpoonright V_{\gamma+1}, j(V_{\gamma+2})\} \in X$. Since a is stationary in $\mathcal{P}(V_{\gamma+1})$ and $a \subset \mathcal{P}_\gamma(V_{\gamma+1})$ we get $|X \cap V_{\gamma+1}| < \gamma$ and $X \cap V_{\gamma+1} = X \cap (\cup a) \in a$, therefore $j(X \cap V_{\gamma+1}) \in j(a)$, given us $j[X \cap V_{\gamma+1}] \notin j(sp(D_\alpha \cap \mathbb{Q}^S_{<\gamma}))$.

Note that since $V_{j(f)(\gamma)+\omega} \subset M$, $X \cap V_{\gamma+1} \in M$ as $j[X \cap V_{\gamma+1}] \in M$ then $j \upharpoonright (X \cap V_{\gamma+1}) \in M$. By the way we chose X, $j[X \cap V_{\gamma+1}] \subset X$.

Let $\leq$ be a well order of $j(V_{\gamma+1)}$ in $M \cap X$ and Y the Skolem closure of $\{a, b\} \cup j \upharpoonright (X \cap V_{\gamma+1}) \cup (X \cap (\cup c))$ in $j(V_{\gamma+1})$, Y can be computed in M, $Y \in M$, and $Y \subset X$, all these sets are subsets of X.

Clearly $j(X \cap V_{\gamma+1}) \subset Y$. By the way we chose X, $b \in X$ and $X \cap (\cup b) \in b$.

Since γ is closed under f, $j(f)(\gamma) < j(\gamma)$, and by (2), $b \in j(\mathbb{Q}^S_{<\gamma})$ and $b \in j(D_\alpha) \cap M_{j(f)(\gamma)}$, then $b \in j(\mathbb{Q}^S_{<\gamma} \cap D_\alpha)$, so $b \in Y \cap j(\mathbb{Q}^S_{<\gamma} \cap D_\alpha)$ and $X \cap (\cup c) \cap (\cup b) \subset Y \cap (\cup b)$ this is $X \cap (\cup b) \subset Y \cap (\cup b)$, but $Y \subset X$ so $X \cap (\cup b) = Y \cap (\cup b) \in b$. Since $\gamma = cp(j)$ we get $Y \cap V_\gamma = j(X \cap V_{\gamma+1}) \cap V_\gamma$, so $Y \cap V_\gamma$ end-extend $j(X \cap V_{\gamma+1}) \cap V_\gamma$.
We conclude $j(X \cap V_{\gamma+1}) \in j(sp(D_\alpha \cap \mathbb{Q}^S_{<\gamma}))$ a contradiction. □

Corollary 4.20. *Suppose that ζ is an ordinal, δ is a Woodin cardinal and κ is a limit ordinal such that $\zeta < \delta < cf(\kappa)$. Let $Y \prec V_\kappa$ be countable with $\zeta, \delta \in Y$. Then there exists a countable $Y' \prec V_\kappa$ such that:*

- *$Y \subset Y'$.*
- *$Y' \cap V_\zeta = Y \cap V_\zeta$.*
- *For each predense $D \subset \mathbb{Q}_{<\delta}$ with $D \in Y'$, there exists $d \in D \cap Y'$ such that $Y' \cap (\cup d) \in d$.*

Proof. Let $e : \omega \to \omega$ be such that for every $i \in \omega$, $e(i) \leq i$ and $e^{-1}(i)$ is infinite, and let $e^* : \omega \to \omega$ define by $e^*(i) = |\{j < i : e(j) = e(i)\}|$.
We will build a chain $< Y_i : i < \omega >$ of countable elementary submodels of V_κ, a sequence $< d_i : i < \omega >$ of elements of $\mathbb{Q}_{<\delta}$ and a set $\{D^i_j : i, j < \omega\}$ such that:

1. $Y_0 = Y$.
2. For every $i < \omega$, $\{D^i_j : j < \omega\}$ lists the predense subsets of $\mathbb{Q}_{<\delta}$ in Y_i.
3. For all $i < j < \omega$ $Y_i \subseteq Y_j$.
4. For every $i < \omega$, if ξ is the supremum of ζ and $sup(\bigcup_{j<i}((\cup d_j) \cap \delta))$ then
$$Y_{i+1} \cap V_\xi = Y_i \cap V_\xi$$
5. For every $i < \omega$, $d_i \in D^{e(i)}_{e^*(i)} \cap Y_i$ and $Y_{i+1} \cap (\cup d_i) \in d_i$.

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And finally $Y' = \cup\{Y_i : i < \omega\}$ works. To construct such sets by induction assume Y_i, $\{D_j^{i'} : i' \leq i \wedge j < \omega\}$ and d_j for $j < i$ are given such that they satisfy 1-5. By the theorem 4.19 and the fact 3.10, there exists a strongly inaccessible $\xi > \zeta \cup (\bigcup_{j<i}((\cup d_j) \cap \delta))$ in Y_i such that $D_{e^*(i)}^{e(i)} \cap \mathbb{Q}_{<\xi}$ is semi-proper.

Claim: There exists $Y^* \prec V_{\xi+1}$ containing $Y_i \cap V_{\xi+1}$ such that $Y^* \cap V_\xi$ end-extends $Y_i \cap V_\xi$ and $Y^* \cap (\cup d) \in d$ for some $d \in Y^* \cap D_{e^*(i)}^{e(i)}$.
Proof of the claim: Let F be the function in $V_{\xi+1}$, $F : [V_{\xi+1}]^{<\omega} \to V_{\xi+1}$ such that $C_F \subseteq D_{e^*(i)}^{e(i)} \cap \mathbb{Q}_{<\xi}$, since $Y_i \prec V_\kappa$ then $Y_i \cap V_{\xi+1}$ is closed under F, $Y_i \cap V_{\xi+1} \in sp(D_{e^*(i)}^{e(i)} \cap \mathbb{Q}_{<\xi})$ and let $Y^* \prec V_{\xi+1}$ be the set that testifies this.

Let d_i be the d of the claim and since $Y_i \prec V_\kappa$,

$$Y_{i+1} := \{f(s) : f : V_\xi \to V_\kappa \wedge f \in Y_i \wedge s \in Y^* \cap V_\xi\}$$

is a countable substructure of V_κ and contains Y_i, so $j \leq i$, $Y_{i+1} \cap (\cup d_j) \in d_j$ and $Y_{i+1} \cap V_\xi \subseteq Y^* \cap V_\xi = Y_i \cap V_\xi \subseteq Y_{i+1} \cap V_\xi$. □

The following remark relates the previous corollary with the set a defined in the theorem 4.15.

Remark. The set a of the lemma 4.15 is stationary when κ_1 is a Woodin cardinal. If δ is a Woodin cardinal, $\zeta < \delta$ and $\delta < \kappa$ a cardinal such that $\delta < cf(\kappa)$. Let F be a function $F : V_{\delta+1}^{<\omega} \to V_{\delta+1}$. Therefore there exists $g : V_{\delta+1}^{<\omega} \to V_{\delta+1}$ such that $C_g \subseteq C_F$ and if $Y \in C_g$ then $Y \prec V_{\delta+1}$. Let X be countable such that $X \prec V_\kappa$, and $g, \zeta, \delta \in X$. By the corollary 4.20, there exists $X' \prec V_\kappa$ with $X' \cap V_{\delta+1} \in a$ and $X \subseteq X'$, therefore $g \in X'$ and $X' \cap V_{\delta+1} \in C_g$.

Theorem 4.21. *Suppose δ is a strongly inaccessible cardinal, and let $\eta < \delta$ such that for each sequence $< D_\alpha : \alpha < \eta >$ of predense subsets of $\mathbb{Q}_{<\delta}^S$ there exists a strongly inaccessible cardinal $\gamma < \delta$ such that for every $\alpha < \eta$, $D_\alpha \cap \mathbb{Q}_{<\gamma}^S$ is*

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predense and semi-proper. Let G be a V-generic contained in $\mathbb{Q}^S_{<\delta}$, then (M, E) is closed under sequences of length η in $V[G]$.

Proof. Fix $a_0 \in \mathbb{Q}^S_{<\delta}$ and a term τ for an η-sequence of (M, E)-ordinals. For each $\alpha < \eta$, choose a maximal antichain $A_\alpha \subseteq \mathbb{Q}^S_{<\delta}$ such that for each $b \in A_\alpha$ there exists $f \in Ord^b$ such that $b \Vdash \tau(\check{\alpha}) \sim [\check{f}]_G$. By fact 3.10, there exists a strongly inaccessible cardinal $\gamma < \delta$ such that

- $a_0, \eta \in V_\gamma$.
- $\forall\alpha < \gamma$, $A_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is predense and semi-proper.

Let a be the set of sets $X \prec V_{\gamma+1}$ such that

1. $|X| < min\{\gamma, \lambda\}$.
2. $X \cap (\cup a_0) \in a_0$.
3. for every $\alpha \in X \cap \eta$ there exists $b \in X \cap A_\alpha \cap \mathbb{Q}^S_{<\gamma}$ such that $X \cap (\cup b) \in b$.

Claim: $a \in \mathbb{Q}^S_{<\delta}$

Proof of the claim: It is enough to show that a is stationary in $\mathcal{P}(V_{\gamma+1})$. Fix $H : [V_{\gamma+1}]^{<\omega} \to V_{\gamma+1}$. Since a_0 is stationary, $a_0^{V_\delta}$ is stationary, by lemma 2.9 and lemma 2.15 we can choose $X_0 \in a_0^{V_\delta}$ such that $X_0 \prec V_\gamma$, $|X_0| = \omega$, and $a_0, H, \gamma, < A_\alpha \cap \mathbb{Q}^S_{<\gamma} : \alpha < \eta >\in X_0$ and $\{A_\alpha \cap \mathbb{Q}^S_{<\gamma} : \alpha \in X_0 \cap \eta\} \subseteq X_0$. Define an elementary chain $< X_\alpha : \alpha \in X_0 \cap \eta >$ as follows, for $\beta \in X_0 \cap \eta$ a limit ordinal $X_\beta = \cup\{X_\alpha : \alpha < \beta, \alpha \in X_0 \cap \eta\}$. Let $\alpha \in X_0 \cap \eta$ with X_α given such that $X_\alpha \prec V_\delta$ and $|X_\alpha| < \min\{\gamma, \lambda\}$. Since $A_\alpha \cap \mathbb{Q}^S_{<\gamma}$ is semi-proper then there exists $F : [V_{\gamma+1}]^{<\omega} \to V_{\gamma+1}$ such that $C_F \subseteq A_\alpha \cap \mathbb{Q}^S_{<\gamma}$, since $X_\alpha \prec V_\delta$ and $A_\alpha \cap \mathbb{Q}^S_{<\gamma} \in X_\alpha$ therefore $X_\alpha \cap V_{\gamma+1} \in sp(A_\alpha \cap \mathbb{Q}^S_{<\gamma})$. Let $Y \prec V_{\gamma+1}$ such that $X_\alpha \cap V_{\gamma+1} \subseteq Y$, $Y \cap V_\gamma$ end-extends $X_\alpha \cap V_\gamma$ and for some $d \in Y \cap X_\alpha \cap V_{\gamma+1}$, $Y \cap (\cup d) \in d$; since $X_\alpha \prec V_\delta$,

$$X_{\alpha+1} = \{f(x) |\ f : V_\gamma \to V_\delta \wedge f \in X_\alpha \wedge x \in V_\gamma \cap Y\}$$

is an elementary substructure of V_δ, $X_{\alpha+1} \cap V_\gamma = Y \cap V_\gamma$ and $X_{\alpha+1} \cap (\cup b) \in b$ for some $b \in A_\alpha \cap \mathbb{Q}^S_{<\gamma} \cap X_{\alpha+1}$.

Let $X = \bigcup\{X_\alpha : \alpha \in X_0 \cap \eta\}$. Since $|X_0| = \omega$ and for every $\alpha \in X_0 \cap \eta$, $|X_\alpha| < min\{\gamma, \lambda\}$ then X satisfies (1).

We have $X \prec V_\delta$ which implies $X \cap V_{\gamma+1}$ is closed under H.

Note that for every $\alpha < \beta$, with $\alpha, \beta \in X_0 \cap \eta$ we have $X_\alpha \cap \eta = X_0 \cap \eta$ and $X_\alpha \cap V_\gamma$ end-extends $X_\beta \cap V_\gamma$, so $X \cap \eta = X_0 \cap \eta$ and $X \cap V_\gamma$ end-extends $X_0 \cap V_\gamma$, which means that $X \cap V_{\gamma+1}$ satisfies (2). And for each $\alpha \in X \cap \eta$ there exists $b \in X_{\alpha+1} \cap A_\alpha \cap \mathbb{Q}^S_{<\gamma}$ such that $X_{\alpha+1} \cap (\cup b) \in b$ but $X_{\alpha+1} \cap (\cup b) = X \cap (\cup b)$ so X satisfies (3). We conclude that a is stationary.

Note that by the definition of a, $a \leq a_0$, so the set of conditions in $\mathbb{Q}^S_{<\delta}$ with elements $X \prec V_{\gamma+1}$ and that satisfies (1) and (3), is a dense set. So we can assume $a \in G$ for some a_0.

By the properties of a, for each $X \in a$ and $\alpha \in X \cap \eta$ there exists $b \in X \cap A_\alpha \cap \mathbb{Q}^S_{<\gamma}$ such that $X \cap (\cup b) \in b$. Since $b \in A_\alpha$, by the way we defined A_α there is a function $f_{(b,\alpha)} \in Ord^b$ such that $b \Vdash [\check{f}_{(b,\alpha)}]_G \sim \tau(\check{\alpha})$. Define f as a function from a to V such that $f(X)(\alpha) = f_{(b,\alpha)}(X \cap (\cup b))$, $f(X)$ is a function with domain $X \cap \eta$; thus $[f]$ is a function in (M, E) with domain $j[\eta]$.

Fix $\alpha < \eta$ and $a_\alpha = \{X \in a : \alpha \in X\}$, as in Lemma 4.15, $a_\alpha \in G$. Let $b \in G \cap A_\alpha$, then there exists $c \in G$ such that $c \leq a_\alpha$, $c \leq b$ so $Y \in c$, $Y \cap (\cup a_\alpha) \in a_\alpha$ therefore $\alpha \in Y$ and there exists $d \in A_\alpha \cap \mathbb{Q}^S_{<\delta} \cap Y \cap (\cup a_\alpha)$ such that $Y \cap (\cup a_\alpha) \cap (\cup d) \in d$, but $d \in A_\alpha \cap \mathbb{Q}^S_{<\delta}$ then $\cup d \subseteq \cup a_\alpha \subseteq \cup c$ and $Y \cap (\cup a_\alpha) \cap (\cup d) = Y \cap (\cup d)$.

Define $F : c \to A_\alpha$, $F(Y) = d$ with $d \in A_\alpha \cap \mathbb{Q}^S_{<\delta} \cap Y$ and $Y \cap (\cup d) \in d$; by normality there exists $d \in A_\alpha$ such that $c' = \{Y \in c : F(Y) = d\} \in \mathbb{Q}^S_{<\delta}$ and $c' \leq d$, but $c' \leq b$, we conclude $d = b$.

Therefore $a \Vdash [\check{f}]_G(j(\check{\alpha})) = \tau(\check{\alpha})$. We conclude that f represents in (M, E) the function from $j[\eta]$ to M given by $j(\alpha) \to \tau_G(\alpha)$, $\tau_G \in M$. □

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Theorem 4.22. *Suppose δ is a Woodin cardinal. Let G be a V-generic, then (M, E) is wellfounded and $M^{<\delta} \subset M$ in $V[G]$.*

Proof. Follows from theorem 4.19 and 4.21. □

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Chapter 5

Applications

" *Don't cry because it's over, smile because it happened.*

- Dr. Seuss "

5.1 Generic Absoluteness

In Namba forcing the cofinality of ω_2 is change to ω without collapsing ω_1; using the stationary tower this can be generalized in such a way that the cofinality of λ a regular cardinal can be changed to any other regular cardinal without collapsing the cardinals below λ.

Example 5.1. *Let δ be a Woodin cardinal and $\gamma < \lambda < \delta$ regular cardinals. The set $a = \{\alpha < \lambda : cf(\alpha) = \gamma\}$ is stationary in $\mathcal{P}(\lambda)$, $a \in \mathbb{P}_\delta$. Suppose G is a Vgeneric such that $a \in G$ and let j be the associated generic embedding, by corollary 3.14 $j[\cup a] \in j(a)$, $j[\lambda] \in \{\alpha < j(\lambda) : cf(\alpha) = j(\gamma)\}$ so $j[\lambda] \in j(\lambda)$, thus $\lambda \in G$ and since λ is regular $cp(j) = \lambda$. We conclude $cf(\lambda) = j(\gamma) = \gamma$ in M. Since $M^{<\delta} \subseteq M$ in $V[G]$, $cf(\lambda) = j(\gamma) = \gamma$ in $V[G]$ and the cardinals below λ are preserve.*

The first applications of the stationary tower were in absoluteness results, the first one that we are going to show is about the theory of Chang models.

Definition 5.1. *We call $L(Ord^{\omega})$ the Chang model and it is defined as*

$$L(Ord^{\omega}) = \bigcup_{\alpha \in Ord} L(\mathcal{P}_{\omega_1}(\alpha))$$

Definition 5.2. *If M is a set (class) such that every member of X has rank less than some ordinal in M. We let*

$$M(X) = \bigcup_{\alpha \in M \cap Ord} \bigcup_{\beta \in M \cap \alpha} L_{\alpha}(M \cap V_{\beta}, X \cap V_{\beta})$$

A set of ω-sequences of ordinals σ is closed under finite sequences if for each finite subset a of σ there is a $z \in \sigma$ such that z recursively codes each member of a.

In a transitive model M of ZF, a set x is generic over M if it exists in a generic extension of M, x induces a generic extension and we call $M[x]$ the minimal extension of M that contains x as an element.

Definition 5.3. *Let M be a transitive model of ZFC, σ a set (class) of ω-sequences of ordinals, each generic over M and closed under finite sequences, and either*

$$Ord^{\omega} \cap M(\sigma) = \sigma$$

or

$$\rho^{\omega} \cap M(\sigma) = \sigma$$

for some ordinal ρ. Let δ be an ordinal in M such that M_{δ} has cardinality δ in M. $M(\sigma)$ is a symmetric extension of M for $coll(\omega, < \delta)$ if in some set generic extension of M there exists an M-generic filter $G \subset coll(\omega, < \delta)$ such that

$$\sigma = \bigcup \{Ord^{\omega} \cap M[G \restriction \alpha] : \alpha < \delta\}$$

or

$$\sigma = \bigcup \{\rho^{\omega} \cap M[G \restriction \alpha] : \alpha < \delta\}$$

The following results will be used in the proof of lemma 5.4, the proofs can be found in [Lar04] pp 124-125.

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- If M is a transitive set (or class) model of ZFC, κ is a limit ordinal in M and x and y are sets such that $\{x, y\}$ exists in a generic extension of M by forcing in M_κ, then either $y \in M[x]$ or y exists in a forcing extension of $M[x]$by forcing in $(M[x])_\kappa$.

- If $\mathbb{P}$ and $\mathbb{Q}$ are partial orders such that forcing with $\mathbb{Q}$ makes $(2^{\mathbb{P}})^V$ countable, then there is a $\mathbb{P}-$name τ for a partial order such that $\mathbb{Q} \cong \mathbb{P} * \tau$.

Lemma 5.4. *Let M be a transitive model of ZFC, and δ a strong limit cardinal in M. $\sigma \subseteq Ord^\omega$ a set (class) of countable sequences of ordinals, each generic over M and closed under finite sequences, such that $Ord^\omega \cap M(\sigma) = \sigma$ or $\rho^\omega \cap M(\sigma) = \sigma$ for some ordinal ρ. Then $M(\sigma)$ is a symmetric extension of M for $coll(\omega, < \delta)$ if and only if*

1. *Each $x \in \sigma$ is M-generic for some forcing $\mathbb{P} \in M_\delta$.*

2. *$\delta = sup\{\omega^{M[x]} : x \in \sigma\}$.*

Proof. $\Rightarrow$The definition of symmetric extension gives $\sigma = \{Ord^\omega \cap M[G \restriction \alpha] : \alpha < \delta\}$ which implies (1) and since $G \subset coll(\omega, < \delta)$ we get (2).

$\Leftarrow$ Define $\mathbb{P}$ a partial order consisting of stes g such that for some $\alpha < \delta$, $x \in \sigma$, g is an M-generic filter in $M[x]$ contained in $coll(\omega, < \alpha)$, ordered by extension.

By (1) and (2), $\mathbb{P} \neq \emptyset$, since for every $\alpha < \delta$ there exists $x \in \sigma$ such that α is countable in $M[x]$. And by (1) $\mathbb{P} \in M(\sigma)$.

Let $G_{\mathbb{P}}$ be an $M(\sigma)$-generic for $\mathbb{P}$ and $H = \cup G_{\mathbb{P}}$. Note that $H \subset coll(\omega, < \delta)$, $H \neq \emptyset$.

If $p \leq q$, $p \in H$ then for some $g \in G_{\mathbb{P}}$, $p, q \in g$ so $q \in H$.

If $p, q \in H$ then there exists $g_p, g_q \in G_{\mathbb{P}}$ with $p \in g_p$ and $q \in g_q$, so there exists $g \in G_{\mathbb{P}}$, $g \leq g_p, g_q$ then $p, q \in g$ therefore there exists $r \in g$ such that

$r \leq p, q$.

H is a filter, indeed an M-generic, to prove that, let D be a dense subset of $coll(\omega, < \delta)$ in M and g an M-generic for $coll(\omega, < \eta)$ for some $\eta < \delta$.

Since D is dense, for every $p \in coll(\omega, < \eta)$ there exists $q \in D$ such that $q \leq p$, this implies $q \cap coll(\omega, < \eta) \leq p$, therefore the set $\{q \cap coll(\omega, < \eta) : q \in D\}$ is dense in $coll(\omega, < \eta)$, since g is M-generic in $coll(\omega, < \eta)$, there exists $p \in D$ such that $p \cap coll(\omega, < \eta) \in g$, and there exists $\eta' < \delta$ such that $p \in coll(\omega, < \eta')$. Since δ is strong limit in M, then there exists $x \in \sigma$ and an M-generic $g' \subset coll(\omega, < \eta')$ in $M[x]$ such that $p \in g'$ and $g' \leq g$. We conclude $p \in H$.

$\bigcup\{Ord^{\omega} \cap M[H \restriction \alpha] : \alpha < \delta\} \subseteq \sigma$

Since $H \cap coll(\omega, < \alpha) \in M(\sigma)$ for every $\alpha < \delta$ so $\mathbb{R}^{M[H \cap coll(\omega, < \alpha)]} \subseteq \mathbb{R}^{M(\sigma)} \subseteq Ord^{\omega} \cap M(\sigma) = \sigma$.

$\sigma \subseteq \bigcup\{Ord^{\omega} \cap M[H \restriction \alpha] : \alpha < \delta\}$

We are going to prove that for each $x \in \sigma$ there exists $\alpha < \delta$ such that $x \in M[H \cap coll(\omega, < \alpha)]$. Working in $M(\sigma)$, fix $x \in \sigma$ we are going to show that $D_x = \{g \in \mathbb{P} : x \in M[g]\}$ is dense in $\mathbb{P}$. Let $y \in \sigma$, $g \subset coll(\omega, < \eta)$ an M-generic, $g \in \mathbb{P} \cap M[y]$. If $x \notin M[g]$, since σ is closed under finite sequences then $\{x, y\}$ exists in a generic extension of M forcing in M_{δ}, then x exists in a forcing generic extension of $M[g]$ forcing in $(M[g])_{\eta'}$ for some $\eta' < \delta$.

Choose $z \in \sigma$ such that $M_{\eta'+1}$ is countable in $M[z]$ and x, y are in $M[z]$.

Therefore exists $g' \subset coll(\omega, < \eta')$ an M-generic in $M[z]$ and $g' \leq g$, we conclude $x \in M[g']$, since $x \in M[z]$. □

Lemma 5.5. *Suppose that δ is a Woodin cardinal which is a limit of Woodin cardinal. Let $G \subset \mathbb{Q}_{<\delta}$ be V-generic. Then $V((Ord^{\omega})^{V[G]})$ is a symmetric extension of V for $coll(\omega, < \delta)$.*

Proof. Let $j : V \to M$ be the embedding corresponding to G in $\mathbb{Q}_{<\delta}$. By corollary 4.13 and theorem 4.22 $j(\omega_1) = \delta$,
$sup\{\omega_1^{V[x]} : x \in (Ord^\omega)^{V[G]}\} \in M$, and $\delta = sup\{\omega_1^{V[x]} : x \in (Ord^\omega)^{V[G]}\}$, $(Ord^\omega)^{V[G]}$ satisfies the second condition of lemma 5.4.
By a previous remark $G \cap \mathbb{Q}_{<\delta'}$ is V-generic when δ' is a Woodin cardinal. Therefore the set A of strongly inaccessible cardinals $\lambda < \delta$ such that $G \cap \mathbb{Q}_{<\lambda}$ is V-generic for $\mathbb{Q}_{<\lambda}$, is cofinal below δ. Let $x \in (Ord^\omega)^{V[G]}$, then there exists A_i $(i < \omega)$ a maximal $\mathbb{Q}_{<\delta}$-antichain in V each one deciding the i-th element of x. And for each A_i there exists $\lambda < \delta$ such that $G \cap A_i \cap \mathbb{Q}_{<\lambda} \neq \emptyset$, since $\delta = \omega_1^{V[G]}$ and A is cofinal, there exists $\lambda < \delta$ such that for every $i < \omega$, $G \cap A_i \cap \mathbb{Q}_{<\lambda} \neq \emptyset$ so $x \in V[G \cap \mathbb{Q}_{<\lambda}]$ and is V-genric, satisfying the first condition of lemma 5.4.
The lemma 5.4 finish the proof. □

The following result can be found in [Rae10] (Proposition 2.54) and is used in the proof of theorem 5.6 (second paragraph).
Let $\mathbb{P}$ be a partial order, G a generic filter on $\mathbb{P}$ and Y a set of ordinals in $V[G]$. For every $p \in \mathbb{P}$ the statement $p \in \mathbb{P}/\dot{Y}$ is equivalent to the existence of a $\mathbb{P}$-generic filter H over V with $\dot{Y}^H = Y$ and $p \in H$.

Theorem 5.6. *Suppose that δ is a Woodin cardinal which is a limit of Woodin cardinals. Then for every H V-generic contained in $coll(\omega, < \delta)$, there exists an elementary embedding $j : L(Ord^\omega)^V \to L(Ord^\omega)^{V[H]}$*

Proof. Let $G \subset \mathbb{Q}_{<\delta}$ be V-generic and $j_G : V \to M$ the corresponding embedding. By lemma 5.5 $V((Ord^\omega)^{V[G]})$ is a symmetric extension of V for $coll(\omega, < \delta)$, so there is a V-generic filter H' contained in $coll(\omega, < \delta)$ such that $(Ord^\omega)^{V[G]} = (Ord^\omega)^{V(\sigma)} = (Ord^\omega)^{V[H']}$ (here $\sigma = (Ord^\omega)^{V[G]}$), and notice that by theorem 4.22 $(Ord^\omega)^{V[G]} = (Ord^\omega)^M$ in $V[G]$, so
$L(Ord^\omega)^{V[H']} = L(Ord^\omega)^M$.
Therefore the restriction $j_G : L(Ord^\omega)^V \to L(Ord^\omega)^M = L(Ord^\omega)^{V[H']}$ is ele-

mentary.

In the proof of the lemma 5.4 H' was defined from a generic $G_{\mathbb{P}}$ with the forcing $\mathbb{P}$, forcing in $V[G]$, so H' is in $V[G][G_{\mathbb{P}}]$, therefore there exists a generic h in $V[H']$ such that $V[H'][h] = V[G][G_{\mathbb{P}}]$.

Consider the formula $\varphi =$"There is a generic h and in the generic extension there is a parameter g, such that $\psi(x, y, g)$ defines an elementary embedding." where $\psi(x, y, g)$ says that the generic embedding of $L(Ord^{\omega})^V$ associated to g maps x into y. Then $V[H'] \models \varphi$ and there exists $p \Vdash^{V[H']}_{coll(\omega,<\delta)} \varphi$, since $coll(\omega, < \delta)$ is weakly homogeneous then $\mathbb{1} \Vdash_{coll(\omega,<\delta)} \varphi$ so
$V[H] \models \varphi$. □

The following theorem is due to Solovay, [Sol70], the proof is also in [Kan03].

Theorem 5.7 (Solovay). *Suppose that κ is an inaccessible cardinal and G is a V-generic contained in $coll(\omega, < \kappa)$. Then in $V[G]$, every set of reals definable from a countable sequence of ordinals is Lebesgue measurable, has the Baire property and has the perfect set property.*

Corollary 5.8. *If there exists a Woodin cardinal which is a limit of Woodin cardinals, then every set of reals in the Chang model is Lebesgue measurable, has the Baire property and has the perfect set property.*

Proof. Let δ be a Woodin cardinal which is a limit of Woodin cardinals and G a V-generic contained in $coll(\omega, < \delta)$. By theorem 5.7, every set of reals in $L(Ord^{\omega})^{V[G]}$ is Lebesgue measurable, has the Baire property and has the perfect set property, but by theorem 5.6, there exists an elementary embedding $j : L(Ord^{\omega})^V \to L(Ord^{\omega})^{V[G]}$, so every set of reals in $L(Ord^{\omega})^V$ is Lebesgue measurable, has the Baire property and has the perfect set property. □

Suppose δ is a Woodin cardinal and $\kappa > \delta$ is a strongly inaccessible. By a previous remark we know that the set a defined in the lemma 4.15 is stationary, but it is also compatible with all the conditions of $\mathbb{Q}_{<\delta}$. To show

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this it is enough to show that for every $d \in \mathbb{Q}_{<\delta}$ the set $a \cap d^{V_{\delta+1}}$ is stationary, following the same argument we can choose $\zeta < \delta$ as a cardinal with $d \in V_\zeta$, and for every $F : V_{\delta+1}^{\omega} \to V_{\delta+1}$ we choose g and X in the same way but with the extra assumption $X \cap V_\zeta \in d^{V_\zeta}$, this is possible because d is stationary and X was just an element of a club, now by the corollary 4.20 we know that there exists $X' \prec V_\kappa$ such that $X' \cap V_{\delta+1} \in a$, $X' \cap V_{\delta+1} \in C_F$ and $X' \cap V_\zeta = X \cap V_\zeta \in d^{V_\zeta}$, $X' \cap V_{\delta+1} \in d^{V_{\delta+1}}$.
The condition a is compatible in $\mathbb{P}_{<\kappa}$ with every condition of $\mathbb{Q}_{<\delta}$, such that if $G \subset \mathbb{P}_{<\kappa}$ is a V-generic filter with $a \in G$, then $G \cap \mathbb{Q}_{<\delta}$ is a V-generic filter for $\mathbb{Q}_{<\delta}$, and $j(\omega_1) = \delta$ where $j : V \to (M, E)$ is the embedding corresponding to G. Finally if $j' : V \to N$ is the corresponding embedding to $G \cap \mathbb{Q}_{<\delta}$, then the elementary embedding $k : N \to (M, E)$ given by $k([f]_{G \cap \mathbb{Q}_{<\delta}}) = [f]_G$ is such that $j = k \circ j'$.

Theorem 5.9. *Under the assumption of the continuum hypothesis. Suppose there are class many Woodin cardinals. For every Woodin cardinal δ, being $G \subset \mathbb{Q}_{<\delta}$ a V-generic filter, with induced embedding $j : V \to M$. Then every real in M satisfies the same Σ_1^2 formulas in M and in $V[G]$.*

Proof. Lets proceed by contradiction. Assume there exists a binary formula φ with quantifiers over the reals, $G \subset \mathbb{Q}_{<\delta}$ a V-generic filter, τ a $\mathbb{Q}_{<\delta}$-name for a set of reals, $b \in G$ a condition in $\mathbb{Q}_{<\delta}$, $f : a \to \mathbb{R}$ a function such that b forces that $\varphi([f]_G, \tau_G)$ holds in $V[G]$ but $\exists A \subset \mathbb{R} \varphi([f]_G, A)$ does not hold in M. Let δ' be a Woodin cardinal bigger than δ.
By the previous comments we know that there exists a condition $b \in \mathbb{P}_{<\delta'}$, such that there exists $G' \subset \mathbb{P}_{<\delta'}$ a V-generic that satisfies $G = G' \cap \mathbb{Q}_{<\delta}$ and $j'(\omega_1) = \delta$, where j' is the associated embedding $j' : V \to M'$.
Since $M^{<\delta_1} \subset M$ in $V[G']$ then $\tau_G \in M$ and $\varphi([f]_G, \tau_G)$ holds in M', but $j(\omega_1) = j'(\omega_1)$ and CH holds in V, therefore M and M' have the same reals, so $\exists A \subset \mathbb{R} \varphi([f]_G, A)$ holds in M. The other direction is trivial. □

5.2 Forcing axioms

In [Woo99] the reader can find many applications of the stationary tower, specially for $\mathbb{P}_{<\kappa}$ and $\mathbb{Q}_{<\kappa}$. We are going to finish this chapter by showing an relation between the stationary towers and the forcing axioms.

Lemma 5.10. *Suppose that δ is a Woodin cardinal, $\lambda = \delta$ a regular cardinal and $S = \mathcal{P}_\lambda(V_\delta)$. Let $G \subset \mathbb{Q}^S_{<\delta} = \mathbb{P}_{<\delta}$ be a V-generic and $j : V \to (M, E)$ the associated elementary embedding. Then $j(V_\delta) \subseteq V_\delta[G]$, where*

$$V_\delta[G] = \bigcup_{\alpha<\delta} L_\delta(V_\alpha, G \cap V_\alpha)$$

Proof. Claim: For every $\alpha < \delta$ there exists a cardinal $\alpha < \beta < \delta$ such that $j(\beta) = \beta$.

Proof of the claim: Let $a \in \mathbb{P}_{<\delta}$, by lifting $a' = a^{V_\gamma} \leq a$ for some $\alpha \leq \gamma$, with $\cup a \subseteq V_\gamma$, since δ is a completely Jónsson cardinal β such that $\gamma, a' \in V_\beta$, thus $\{Z \subseteq V_\beta : |Z \cap \beta| = \beta \wedge Z \cap V_\gamma \in a'\}$ is stationary in V_β and stronger than a, therefore the set

$$\{b \in \mathbb{P}_{<\delta} : b \Vdash \exists \beta \in (\alpha, \delta)\ j(\beta) = \beta\}$$

is dense in $\mathbb{P}_{<\delta}$ and the claim follows by genericity.

Let $t \in j(V_\delta)$, by the claim we know that there exists $\gamma < \delta$ a completely Jónsson cardinal such that $t \in j(V_\gamma)$, therefore $cp(j^\infty_{V_\gamma}) > \gamma$ and $t \in j_{V_\gamma}(V_\gamma)$, but by the corollary 3.14 U_{V_γ} is computed from $G \cap V_{\gamma+2}$ then we conclude that $t \in L_\delta(V_{\gamma+2}, G \cap V_{\gamma+2})$. □

Lemma 5.11. *Suppose δ is a Woodin cardinal and λ is a regular uncountable cardinal. Let $S = \mathcal{P}_\lambda(V_\delta)$, G a V-generic and M the generic ultrapower.*
Then $V_\delta[G] = H^{V[G]}_\delta \subseteq H^M_\delta$.

Proof. Claim: $H^{V[G]}_\delta \subseteq H^M_\delta$.
Proof of the claim: Working in $V[G]$. Suppose $x \in H^{V[G]}_\delta$ is transitive. Let α

be a cardinal and $E \subset \alpha \times \alpha$, such that $(\alpha, E) \equiv (x, \in)$.
Let $h : \alpha \times \alpha \to \alpha$ denote the Gödel pairing, and $f : \alpha \to V$, a function such that $f(\beta) = 1$ if and only if $h^{-1}(\beta) \in E$. Then since $M^{<\delta} \subset M$, $f \in M$ and $x \in M$, since $(\beta, \gamma) \in E \Leftrightarrow f(h(\beta, \gamma)) = 1$.

Claim: $V_\delta[G] = H_\delta^{V[G]}$.
Proof of the claim: Suppose $x \in V_\delta[G]$, so exists $\alpha, \gamma < \delta$ such that $x \in L_\gamma(V_\alpha, G \cap V_\alpha)$. So it is enough to show $L_\gamma(V_\alpha, G \cap V_\alpha) \in H_\delta^{V[G]}$ for every $\gamma, \alpha < \delta$.
To prove this, lets do an induction over γ to prove that

- $L_\gamma(V_\alpha, G \cap V_\alpha) < \delta$.
- $\forall x \in L_\gamma(V_\alpha, G \cap V_\alpha)$: $|x| < \delta$.

From this follows $L_\gamma(V_\alpha, G \cap V_\alpha) \in H_\delta^{V[G]}$.
If $\gamma = 0$ it is easy to see that since δ is inaccessible in V, $V_\alpha \in H_\delta$ and since $G \cap V_\alpha \subseteq V_\alpha$, γ satisfies the inequalities. For the successor step, note that since δ is inaccessible in V, for every $\gamma < \delta$, $|L_\gamma(V_\alpha, G \cap V_\alpha)| \leq \max(\{|V_\alpha|, \omega\}) \cdot \gamma$. Thus $L_{\gamma+1}(V_\alpha, G \cap V_\alpha) < \delta$ and for every x in $L_{\gamma+1}(V_\alpha, G \cap V_\alpha)$, $|x| < \delta$. The limit case follows from the successor step.
For the other direction let $x \in H_\delta^{V[G]}$, x is determined by less than λ antichains, each one meeting G in a V_α for $\alpha < \delta$ (in V), $x \in \bigcup_{\alpha < \delta} L_\delta(V_\alpha, G \cap V_\alpha)$.□

Note that when $\lambda = \delta$ the lemma 5.11 is the other inclusion of the lemma 5.10 and the equality holds in the lemma 5.11 too. To show the equality of 5.11 we argument that since δ is inaccessible in V, $j(\delta)$ is inaccessible in M so $V_{j(\delta)}^M = H_{j(\delta)}^M$. Therefore since $H_\delta^M \subseteq H_{j(\delta)}^M$, $H_\delta^M \subseteq V_{j(\delta)}^M$. Since j is elementary, $H_\delta^M \subseteq j(V_\delta)$ and finally since $\lambda = \delta$, by lemma 5.10 $H_\delta^M \subseteq V_\delta[G]$.

Lemma 5.12 (Viale). *Let δ, λ, S and G be as in the lemma 5.11, and j the generic embedding. Then for every $\gamma < cp(j)$, $\mathbb{Q}_{<\delta}^S$ preserve the stationary subsets of γ.*

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Proof. Let a be a stationary subset of γ, G a V-generic for $\mathbb{Q}^S_{<\delta}$ and $\dot{C}$ a $\mathbb{Q}^S_{<\delta}$-name for a club subset of γ. Then $\dot{C}^G \in H_\gamma^{V[G]}$, by the lemma 5.11 $\dot{C}^G \in H_\gamma^M$ and there exists a subset $X \subset V_\delta$ and a function $f : S^X \to X$ such that $[f]_{U_X} = \dot{C}^G$, therefore $\{x \in S^X : f(x) \text{ is a club}\} \in U_X$ and $f(x) \cap a \neq \emptyset$ for all these x, thus $\{x \in S^X : f(x) \cap a \neq \emptyset\} \in U_X$. We conclude
$M \models [f] \cap j(a) \neq \emptyset$, and since $j(\gamma) = \gamma$, $j(a) = a$ and $M \models [f] \cap a \neq \emptyset$. □

Note that the towers $\mathbb{R}^\lambda_\delta$ preserve the stationary subsets of γ for every $\gamma < \lambda$, since $cp(j) = \lambda$ for every G V-generic.
Before we state the next property of the stationary tower we need to fix some notation and to mention some concepts.
Given two forcing notions $\mathbb{P}$, $\mathbb{Q}$. We say that $\mathbb{P}$ completely embeds into $\mathbb{Q}$ if there exists a map $i : \mathbb{P} \to \mathbb{Q}$ such that

- $i(\mathbb{1}_\mathbb{P}) = \mathbb{1}_\mathbb{Q}$.
- $p_1 \leq_\mathbb{P} p_2$ implies $i(p_1) \leq_\mathbb{Q} i(p_2)$.
- p_1 and p_2 are compatible if and only if $i(p_1)$ and $i(p_2)$ are compatible.
- If A is a maximal antichain in $\mathbb{P}$ then $i(A)$ is a maximal antichain.

For a forcing notion $\mathbb{P}$ and a condition $p \in \mathbb{P}$, we denote by $\mathbb{P} \upharpoonright p$ the set $\{(x,y) \in \mathbb{P} : x < p\}$, the restriction of $\mathbb{P}$ to the conditions below p. $\mathbb{B}(\mathbb{P})$ denotes the complete boolean algebra such that $\mathbb{P}$ can be embeds in $\mathbb{B}(\mathbb{P})$ by a dense embedding. For more about forcing with complete boolean algebras, the reader can check [Jec03] or [Kun11], the respective section in the forcing chapter. Finally, for a partial order $\mathbb{P}$, κ an uncountable regular cardinal and $X \prec H_\kappa$, an X-generic filter for $\mathbb{P}$ is a filter G that intersects every dense subset of $\mathbb{P}$, $D \in X$, in X, i.e. $D \cap G \cap X \neq \emptyset$.

Theorem 5.13. *Assume there are class many Woodin cardinals. Let λ be a regular cardinal, and $\mathbb{P}$ a partial order. Then the following are equivalent:*

1. *$\{X \prec H_{|\mathbb{P}|^+} : |X| < \lambda$ and there is an X-generic filter for $\mathbb{P}\}$ is stationary.*

2. *$\mathbb{P}$ completely embeds into $\mathbb{Q}^S_{<\delta} \restriction b$ for some Woodin cardinal δ, $S = \mathcal{P}_\lambda(V_\delta)$ and some stationary set $b \in \mathbb{Q}^S_{<\delta}$.*

Proof. $1 \to 2$. For each $X \in \{X \prec H_{|\mathbb{P}|^+} : |X| < \lambda\}$ and there is an X-generic filter for $\mathbb{P}\} = a$ let $F_X \in V$ be an X-generic filter for $\mathbb{P}$. Let δ be a Woodin cardinal such that $|\mathbb{P}|, \lambda < \delta$ and $S = \mathcal{P}_\lambda(V_\delta)$. Define $g : S^{\cup a} \to \cup a$ as $g(X) = F_X$ if $X \in a$ and x_0 (a fixed element of $\cup a$) in other case. Then, G being a generic filter for $\mathbb{Q}^S_{<\delta}$ such that $a \in G$ we get $\{x \in S^{\cup a} : g(x)$ is an x-generic filter for $\mathbb{P}\} \in U_{\cup a}$, therefore $[g]$ is a V-generic filter for $\mathbb{P}$ and

$$i : \mathbb{P} \to \mathbb{B}(\mathbb{Q}^S_{<\delta} \restriction a)$$

$$p \mapsto ||p \in [\dot{g}]||_{\mathbb{B}}$$

is a complete embedding.
$2 \to 1$. Let $a \in \mathbb{Q}^S_{<\delta}$ be a conditions such that $\mathbb{P}$ completely embeds into $\mathbb{Q}^S_{<\delta} \restriction a$. Thus if G is a V-generic such that $a \in G$, $H = i^{-1}(G)$ is a V-generic for $\mathbb{P}$ and $H \in V_\delta[G]$, by lemma 5.11 $H \in M$, thus $H = [f]_{U_X}$ for some $X \subset V_\delta$ and a function $f : S^X \to X$, therefore $b = \{M \in S^X : f(M)$ is an M-generic filter for $\mathbb{P}\} \in U_X$, in particular for some $X \supset H_{|\mathbb{P}|^+}$, so $b_{H_{|\mathbb{P}^+|}}$ is stationary in $H_{|\mathbb{P}^+|}$ and since the sets $M \prec H_{|\mathbb{P}^+|}$ form a club in $H_{|\mathbb{P}^+|}$ we are done. □

Remark. in the previous theorem in 2. if b is compatible with a_λ (a_λ was defined in the corollary 3.18) then $\{X \prec H_{|\mathbb{P}|^+} : X \cap \lambda \in \lambda |X| < \lambda$ and there is an X-generic filter for $\mathbb{P}\}$ is stationary. Therefore

1. $\{X \prec H_{|\mathbb{P}|^+} : X \cap \lambda \in \lambda |X| < \lambda$ and there is an X-generic filter for $\mathbb{P}\}$ is stationary.

2. $\mathbb{P}$ completely embeds into $\mathbb{R}^\lambda_\delta \restriction b$ for some Woodin cardinal δ, $S = \mathcal{P}_\lambda(V_\delta)$ and some stationary set $b \in \mathbb{R}^\lambda_\delta$.

Definition 5.14. *Given a cardinal λ and a partial order $\mathbb{P}$, we say that $FA_\lambda(\mathbb{P})$ holds if for every collection of λ dense subsets of $\mathbb{P}$, there is a filter $G \subset \mathbb{P}$ that intersects every set of the collection.*

The following theorem is a generalization of the theorem 2.53 of [Woo99], this theorem and the previous one give us a relation between forcing axioms and the stationary tower. The reader can find the proof for the special case when $\lambda = \omega_2$ in [Woo99] pp 41; here we are going to show the general case and the proof is due to Matteo Viale in [Via] pp 15.

Theorem 5.15 (Viale). *Let $\lambda = \alpha^+$ be a successor cardinal, $\mathbb{P}$ a partial order such that $(\mathbb{P}, \leq) = (\kappa, \leq)$ for some $\kappa \in Card$ and $\alpha \leq |\mathbb{P}|$. Then the following are equivalent.*

1. $FA_\alpha(\mathbb{P})$.
2. $\{X \prec H_{|\mathbb{P}|^+} : X \cap \lambda \in \lambda, |X| < \lambda$ *and there is an X-generic filter for* $\mathbb{P}\}$ *is stationary.*

Proof. $1 \to 2$. Choose $\theta > \lambda$ such that $\mathbb{P} \in H_\theta$ and $M_0 \prec H_\theta$ such that $\mathbb{P} \in M_0$, $\alpha \subseteq M_0$ and $|M_0| = \alpha$. Therefore by $\mathrm{FA}_\alpha(\mathbb{P})$, there exists a filter H which meets all the dense sets in M_0.
Define M_1 as

$$M_1 = \{a \in H_\theta : \exists \tau \in M_0 \cap V^{\mathbb{P}} \exists q \in H(q \Vdash a = \tau)\}$$

Claim: $M_1 \prec H_\theta$.
Proof of the claim: Let $\varphi(x_0, \ldots, x_n)$ be a first order formula and $a_1, \ldots, a_n \in M_1$ such that $H_\theta \models \exists x \varphi(x, a_1, \ldots, a_n)$. Let $\tau_1, \ldots, \tau_n \in M_0 \cap V^{\mathbb{P}}$ be such that for each i there exists $q_i \in H$ such that $q_i \Vdash a_i = \tau_i$, and

$$\Vdash \exists x \in H_\theta^V \varphi(x, \tau_1, \ldots, \tau_n)^{H_\theta^V}$$

Since $P \in H_\theta$ there exists $\tau \in H_\theta$ such that

$$\Vdash \varphi(\tau, \tau_1, \ldots, \tau_n)^{H_\theta^V} \wedge \tau \in V$$

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By the way we chose M_0, we know that $\tau \in M_0 \cap V^{\mathbb{P}}$ and that there exists $q \in H$ such that $q \Vdash \tau_i = a_i$ and $q \Vdash \tau = a$ for some $a \in H_\theta$. By the definition of M_1, $a \in M_1$ and $H_\theta \models \varphi(a, a_1, \ldots, a_n)$.

Claim: H is M_1-generic.
Proof of the claim: Let $D \in M_1$ be a dense subset of $\mathbb{P}$. There exists $\tau \in M_0$ and $q \in H$ such that $q \Vdash \tau = D$ and $\Vdash \tau$ is a dense subset of $\mathbb{P}$ which belongs to V.
Since $M_0 \prec H_\theta$, there exists $\tau' \in M_0$ such that $\Vdash \tau' \in \tau \cap \dot{H}$,therefore there exists $r \leq q$, $r \in H$ and $p \in \mathbb{P}$ such that $r \Vdash \tau' = p$ and $r \Vdash \tau = D$, $r \Vdash p \in \dot{H} \cap D$ and $p \geq r$, $p \in H \cap D$, and $p \in M_1$.

Since H is a filter and by the definition of M_1 there is a injective function from M_1 into $M_0 \cap V^{\mathbb{P}}$, and $M_0 \subseteq M_1$, we conclude $|M_1| = |M_0| = \alpha$.
Choose $\alpha \leq \beta$, $\beta \in M_1 \cap \lambda$, $\tau \in M_0$ and $q \in H$ such that $\Vdash \tau \in \lambda$ and $q \Vdash \tau = \beta$. Pick a $\mathbb{P}$-name $\varphi_\tau \in V^{\mathbb{P}} \cap M_0$ such that $\Vdash \varphi_\tau : \alpha \to \tau$ is a bijection which belongs to V, and there exists $r \leq q$, $r \in H$ such that $r \Vdash \varphi_\tau = \varphi$ for some $\varphi \in V$. Since $\alpha \subset M_0$ then $\varphi[\alpha] = \beta \subset M_1$ and $M_1 \cap \lambda \in \lambda$.

Given a function $f : [H_\theta]^{<\omega} \to H_\theta$, and $X \prec H_\theta$ such that $\mathbb{P} \in X$, $\alpha \subseteq X$, $|X| = \alpha$ and $X \in C_f$. By the previous claims, there exists $M_1 \prec H_\theta$ such that $M_1 \in C_f$, $M_1 \cap \lambda \in \lambda, |M_1| < \lambda$ and there is an M_1-generic filter for $\mathbb{P}$. Therefore $\{X \prec H_\theta \cap \lambda \in \lambda, |X| < \lambda$ and there is an X-generic filter for $\mathbb{P}\}$ is stationary. Projecting this set in $H_{|\mathbb{P}|^+}$ we obtain that $\{X \prec H_{|\mathbb{P}|^+} : X \cap \lambda \in \lambda, |X| < \lambda$ and there is an X-generic filter for $\mathbb{P}\}$ is stationary.

$2 \to 1$. Suppose there exists $\{D_\beta\}_{\beta<\alpha}$ such that for every filter $G \subset \mathbb{P}$ there exists $\gamma < \alpha$ such that $G \cap D_\gamma$.
Let $F : [H_{|\mathbb{P}|^+}]^{<\omega} \to H_{|\mathbb{P}|^+}$ be a function such that $F(\emptyset) = \alpha$, $F(\alpha) = (D_\beta)_{\beta<\alpha}$

where $(D_\beta)_{\beta<\alpha}$ is an enumeration of the dense sets. Note that $D_\beta \in H_{|\mathbb{P}|^+}$ and the same with the enumeration. Let C be the club such that $X \in C$ implies $X \in C_F$ and $X \prec H_{|\mathbb{P}|^+}$. For every $X \in C$, $\alpha, (D_\beta)_{\beta<\alpha} \in C$, and if $X \cap \lambda$ is transitive we get $\alpha \subset X$, therefore $D_\beta \in X$ for each $\beta < \alpha$ and $C \cap \{X \prec H_{|\mathbb{P}|^+} : X \cap \lambda \in \lambda |X| < \lambda$ and there is an X-generic filter for $\mathbb{P}\} = \emptyset$. □

Corollary 5.16. *Assume there are class many Woodin cardinals. Let $\lambda = \alpha^+$ be a successor cardinal, $\mathbb{P}$ a partial order such that $(\mathbb{P}, \leq) = (\kappa, \leq)$ for some $\kappa \in Card$ and $\alpha \leq |\mathbb{P}|$.*
Then $FA_\alpha(\mathbb{P})$ implies that $\mathbb{P}$ completely embeds into $\mathbb{Q}^S_{<\delta} \upharpoonright b$ for some Woodin cardinal δ, $S = \mathcal{P}_\lambda(V_\delta)$ and some stationary set $b \in \mathbb{Q}^S_{<\delta}$.

Corollary 5.17 (Viale). *Assume there are class many Woodin cardinals. Let $\lambda = \alpha^+$ be a successor cardinal, $\mathbb{P}$ a partial order such that $(\mathbb{P}, \leq) = (\kappa, \leq)$ for some $\kappa \in Card$ and $\alpha \leq |\mathbb{P}|$. Then the following are equivalent.*

1. *$FA_\alpha(\mathbb{P})$.*
2. *$\mathbb{P}$ completely embeds into $\mathbb{R}^\lambda_\delta \upharpoonright b$ for some Woodin cardinal δ, $S = \mathcal{P}_\lambda(V_\delta)$ and some stationary set $b \in \mathbb{R}^\lambda_\delta$.*

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