Σ-Prikry forcings and their iterations

Universitat de Barcelona

Alejandro Poveda

Departament de Matemàtiques i Informàtica

Joint Set Theory seminar of the BIU and the HUJI
13th May - 2020

Partially supported by MECD Grant no FPU15/00026.

Joint work with A. Rinot & D. Sinapova

1. **Sigma-Prikry forcing I: The axioms**, Submitted to Canadian Journal of Mathematics (2019).
2. **Sigma-Prikry forcing II: Iteration Scheme**, Submitted to Journal of Mathematical Logic (2019).

Iteration theorems for successors of regular cardinals

(I) The $<\aleph_0$-support iteration of ccc forcing is also ccc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_0}}(ccc) = MA$ **(Solovay-Tennembaum)**.

(II) Let Γ be the family of well-met, $\aleph_1$-linked and $\aleph_1$-closed forcings. Under the CH, the $<\aleph_1$-support iteration of forcings in Γ is $\aleph_2$-cc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_1}}(\Gamma) := BA$ **(Baumgartner)**

(III) Let Γ be the family of well-met, $\aleph_2$-stationary-cc and $\aleph_1$-closed forcings with exact upper bounds. Under the CH, the $<\aleph_1$-support iteration of members of Γ is $\aleph_2$-stationary-cc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_1}}(\Gamma)$ **(Shelah)**

(IV) Let $\aleph_1 \leq \mathrm{cof}(\kappa) = \kappa$ and Γ be the family of well-met, κ^+-stationary-cc, κ-closed and countably parallel closed forcing. Under $\kappa^{<\kappa} = \kappa$, the iteration of $<\kappa$-supported iteration of members of Γ is κ^+-stationary-cc **Consistency of** $\mathrm{FA}_{2^\kappa}(\Gamma)$ **(CDMMS)**

Goal

Solve problems at the level of singular cardinal and their successors.

Two approaches

1. The approach of Džamonja and Shelah, and CDMMS:
    - Begin with a large cardinal κ.
    - Define a forcing iteration aimed to solve certain problem about κ^+ by appealing to some of the above iterations theorems.
    - At the end singularize κ by appealing to a Prikry-type forcing. The former iteration should anticipate the effect of this Prikry-type forcing.
2. Find an iteration theorem for κ^{++}-length and κ-supported iterations of κ^{++}-cc forcing, when κ is a singular cardinal.

Goal

Solve problems at the level of singular cardinal and their successors.

Strategy

Find an iteration theorem for κ^{++}-length and κ-supported iterations of κ^{++}-cc forcing, when κ is a singular cardinal.

- In the context of successors of regular cardinals there is a vast theory of iterations (Solovay-Tennembaum, Shelah, CDMMS)
- We know that κ^{++}-cc is not strong enough (even for κ regular) to iterate (Rosłanowski, Shelah) and, besides, that one needs to require additional properties (well-metness, κ^+-closedness with exact bounds, etc)
- An additional caveat is that, for κ singular, the κ^+-closedness with exact bounds is usually not available (e.g. let $S \subseteq E^{\kappa^+}_{\mathrm{cof}(\kappa)}$ non-reflecting and $\mathrm{CU}(\kappa^+, S \cup E^{\kappa^+}_{\neq\mathrm{cof}(\kappa)})$. This is $\mathrm{cof}(\kappa)$-closed hence, if $\omega = \mathrm{cof}(\kappa) < \kappa$, it is not even σ-closed.)

Question

So, if we do not have κ^+-closedness with exact bounds, what can we do?

An alternative: The Prikry workaround

An alternative is to look at forcings $\mathbb{P}$ which are "layered-closed". Namely,

1. $\mathbb{P}$ can be written as $\bigcup_{n<\omega} \mathbb{P}_n$, according to some reasonable notion of length (Graded poset, from Lecture #1).
2. The layers $\mathbb{P}_n$ are eventually as closed as we wish. That is, there is $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ a non-decreasing sequence of uncountable regular cardinals such that, for each $n < \omega$,
    - $\mathbb{P}_n$ is κ_n-closed;
    - $\kappa = \sup_{n<\omega} \kappa_n$;
    - $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\check{\kappa}^+$ is not collapsed".

As we showed in the previous lecture, this is the typical situation for many Prikry-type forcings centered on cofinality ω and motivates the Σ-Prikry framework

Revised Strategy

Find an iteration theorem for κ^{++}-length and κ-supported iterations of κ^{++}-cc Prikry-type forcings, when κ is a singular cardinal.

Note

There already exists iteration theorems for Prikry-type forcing due to Magidor & Gitik.

Let us recall them

Magidor & Gitik iterations

Definition (Gitik)

A set P with two partial orders $\leq$ and $\leq^*$ is called a Prikry-type forcing if $\leq^* \subseteq \leq$ and $\langle P, \leq, \leq^* \rangle$ has the Prikry property; i.e., for each sentence φ in the language of $\mathbb{P} := \langle P, \leq \rangle$-names and each $p \in P$, there is $q \leq^* p$ such that $q \parallel \varphi$.

Magidor iterations (Magidor, Gitik)

Let ϱ be an ordinal. A Magidor iteration of Prikry forcings with length ϱ, $\langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\beta \mid \alpha \leq \varrho, \beta < \varrho \rangle$, is defined by induction as follows. For each $\alpha < \varrho$ we define $\mathbb{P}_\alpha$ to be the set of all sequences $p = \langle p_\beta \mid \beta < \alpha \rangle$ so that, for every $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ and

$$p \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“} p_\beta \in \mathbb{Q}_\beta \ \& \ \langle \dot{Q}_\beta, \dot{\leq}_\beta, \dot{\leq}^*_\beta \rangle \text{ is a Prikry-type forcing”.}$$

Magidor iterations (Magidor, Gitik)

Let ϱ be an ordinal. A Magidor iteration of Prikry forcings with length ϱ, $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\beta \mid \alpha \leq \varrho, \beta < \varrho \rangle$, is defined by induction as follows. For each $\alpha < \varrho$ we define $\mathbb{P}_\alpha$ to be the set of all sequences $p = \langle \dot{p}_\beta \mid \beta < \alpha \rangle$ so that, for every $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ and

$$p \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}p_\beta \in \mathbb{Q}_\beta \ \& \ \langle \dot{Q}_\beta, \dot{\leq}_\beta, \dot{\leq}^*_\beta \rangle \text{ is a Prikry-type forcing”}.$$

Let $p, q \in \mathbb{P}_\alpha$. We write $p \leq_\alpha q$ iff the following are true:

1. For each $\beta < \alpha$, $p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \leq_\beta \dot{q}_\beta$.
2. There is $b \in [\alpha]^{<\aleph_0}$ such that for all $\beta \in \alpha \setminus b$, $p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \leq^*_\beta \dot{q}_\beta$.

Observation

Roughly speaking, the ordering $\leq_\varrho \setminus \leq^*_\rho$ is the $<\aleph_0$-support iteration of the orderings $\leq^*_\alpha$, for $\alpha < \varrho$.

Magidor & Gitik iterations

Theorem (Magidor, Gitik)

The Magidor iteration of Prikry-type forcings is of Prikry-type.

One can define Gitik's iterations in a similar fashion requiring that:

1. Conditions of the iteration have Easton support;
2. $p \leq_\alpha q$ if and only if the following is true:
   1. for each $\beta < \alpha$, $p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \leq_\beta \dot{q}_\beta$;
   2. there is $b \in [\mathrm{supp}(q)]^{<\aleph_0}$ such that for each $\beta \in \alpha \setminus b$, $p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \leq^*_\beta \dot{q}_\beta$.

Remark

Observe that Bullet 2.2. is saying that we are only allowed to modify the stems at finitely $\beta \in \mathrm{supp}(q)$, but still we are free to take non-direct extensions at many α's outside $\mathrm{supp}(q)$.

The utility of Magidor & Gitik iterations

The following lemma illustrates the purpose of Magidor/Gitik iterations:

Lemma (Gitik)

Let κ be a strong compact cardinal and let $\langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be a Magidor iteration of Prikry-type forcing notions such that $\mathbb{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many $\alpha < \kappa$. Besides, assume that the following is true:

1. For every $\alpha < \kappa$, $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\langle \mathbb{Q}_\alpha, \leq^*_\alpha \rangle$ is $|\alpha|$-closed";
2. For all $p, q, r \in \dot{\mathbb{Q}}_\alpha$, if $p, q \leq^* r$ then there is $t \in \dot{\mathbb{Q}}_\alpha$ such that $t \leq^* p, q$.

Then κ is a strong compact cardinal in $V^{\mathbb{P}_\kappa}$.

The utility of Magidor & Gitik iterations

The moral

Magidor & Gitik iterations are, in essence, iterations in the style of Easton.

1. The goal is modify V_κ so that at the end κ enjoys certain property.
2. The chain condition of the iterates grows progressively.
3. The closedness of the orderings $\leq^*_\alpha$ also increases along the iteration.

Some relevant applications

1. Magidor's discovering of the identity crises phenomenon for strong compact cardinals.
2. Gitik & Shelah indestructibility results for strong cardinals.
3. Ben-Neria & Unger result on the existence of an inaccessible cardinal κ joint with a club $C \subseteq \kappa$ where each $\lambda \in C$ is singular and measurable in HOD.

The utility of Magidor & Gitik iterations

The moral

Magidor & Gitik iterations are, in essence, iterations in the style of Easton.

1. The goal is modify V_κ so that at the end κ enjoys certain property.
2. The chain condition of the iterates grows progressively.
3. The closedness of the orderings $\leq^*_\alpha$ also increases along the iteration.

- We want to keep fixed both the chain condition and the degree of "layered-closedness" along the iteration. Thus, we are looking for a different style of iterating Prikry-type forcings.
- In particular, this implies that we need a different abstraction of Prikry-forcings than that given by Gitik. This motivates the Σ-Prikry framework.
- Metaphorically, we aim for something more akin to the iteration that forces $\mathrm{FA}_{2^{\kappa^+}}(\Gamma)$, for κ singular, rather than to the Easton-support iteration that forces $2^\theta = \theta^{++}$ at a measurable cardinal θ.

Iterations of Σ-Prikry forcing

Goal

Solve problems at the level of singular cardinal and their successors.

Strategy

Find an iteration theorem for κ^{++}-length and κ-supported iterations of κ^{++}-cc Prikry-type forcings, when κ is a singular cardinal.

- One of the main features of our iteration is that it is wholly concentrated on the cardinal κ^+.
  - That is, we force at each successor stage $\alpha < \kappa^{++}$ accordingly to destroy a potential counterexample for our intended property at κ^+. The "catch your tail" arguments guarantee that κ^+ enjoys the desired property in the final generic extension.

Iterations of Σ-Prikry forcing

Goal

Solve problems at the level of singular cardinal and their successors.

Strategy

Find an iteration theorem for κ^{++}-length and κ-supported iterations of κ^{++}-cc Prikry-type forcings, when κ is a singular cardinal.

- One of the main features of our iteration is that it is wholly concentrated on the cardinal κ^{+}.
- It is not a forcing iteration in the usual sense.
  - We do not define the successors stages as $\mathbb{P}_{\alpha+1} = \mathbb{P}_{\alpha} * \dot{\mathbb{Q}}_{\alpha}$, where $\dot{\mathbb{Q}}_{\alpha}$ is a $\mathbb{P}_{\alpha}$-name for a forcing notion. Instead we invoke a (ground model) functor $\mathbb{A}(\cdot, \cdot)$ which, given a problem z, produces a forcing $\mathbb{A}(\mathbb{P}_{\alpha}, z)$ solving the problem z and projecting onto $\mathbb{P}_{\alpha}$ (in some strong sense).

Iterations of Σ-Prikry forcing

- It is not a forcing iteration in the usual sense.
  - We do not define the successors stages as $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a forcing notion. Instead we invoke a (ground model) functor $\mathbb{A}(\cdot,\cdot)$ which, given a problem z, produces a forcing $\mathbb{A}(\mathbb{P}_\alpha, z)$ which solves the problem z and projects onto $\mathbb{P}_\alpha$ (in some strong sense).

An advantage of this approach

It allows to keep a good chain conditions even in the presence of $2^\kappa \geq \kappa^{++}$. Observe that in the context of usual iterations, if $\mathbb{P}_\alpha$ forces $2^\kappa \geq \kappa^{++}$, any natural poset devised to add a subset of κ^+ via bounded approximation will not have the κ^{++}-cc in $V^{\mathbb{P}_\alpha}$.

Main theorem (actually a special version when $\mu = \kappa^+$)

Suppose that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of regular uncountable cardinals, converging to a cardinal κ. Let us say that a notion of forcing $\mathbb{P}$ is nice if $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\check{\kappa}^+$ is not collapsed" and $\mathbb{P} \subseteq H_{\kappa^{++}}$. Suppose that:

- $(\mathbb{Q}, \ell, c)$ is a nice Σ-Prikry notion of forcing;
- $\mathbb{A}(\cdot, \cdot)$ is a functor that produces for every nice Σ-Prikry notion of forcing $\mathbb{P}$ and every $\mathbb{P}$-name $z \in H_{\kappa^{++}}$, a corresponding nice Σ-Prikry notion of forcing $(\mathbb{A}(\mathbb{P}, z), \ell', c')$ that admits a forking projection to $\mathbb{P}$ and satisfies some additional properties;
- $2^{2^\kappa} = \kappa^{++}$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++} \rangle$ of elements of $H_{\kappa^{++}}$.

Then there exists a κ-supported sequence $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha) \mid \alpha \leq \kappa^{++} \rangle$ of nice Σ-Prikry forcings such that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$, $\mathbb{P}_{\alpha+1} = \mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$ and, for every pair $\alpha \leq \beta < \kappa^{++}$, $(\mathbb{P}_\beta, \ell_\beta, c_\beta)$ forking projects onto $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ and $(\mathbb{P}_{\kappa^{++}}, \ell_{\kappa^{++}})$ forking projects onto $(\mathbb{P}_\beta, \ell_\beta)$.

The Σ-Prikry framework

1. $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}$;
2. $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals with $\kappa := \sup_{n<\omega} \kappa_n$;
3. μ is a cardinal such that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$;
4. $\ell : P \to \omega$ and $c : P \to \mu$ are functions;

Definition (Σ-Prikry forcing)

We say that $(\mathbb{P}, \ell, c)$ is Σ-Prikry iff all of the following hold:

1. $(\mathbb{P}, \ell)$ is a graded poset;
2. For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$ is κ_n-directed-closed;
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P_0^p \cap P_0^q$ is non-empty;
4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{r \leq^n p \mid q \leq^m r\}$ contains a $\leq$-largest condition $m(p, q)$. In the particular case that $m = 0$, we write $w(p, q)$ instead of $m(p, q)$;

Definition (continuation)

We say that $(\mathbb{P}, \ell, c)$ is Σ-Prikry iff all of the following hold:

1. $(\mathbb{P}, \ell)$ is a graded poset;
2. For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$ is κ_n-directed-closed;
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P_0^p \cap P_0^q$ is non-empty;
4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{r \leq^n p \mid q \leq^m r\}$ contains a $\leq$-largest condition $m(p, q)$. In the particular case that $m = 0$, we write $w(p, q)$ instead of $m(p, q)$;
5. For all $p \in P$, the set $W(p) := \{w(p, q) \mid q \leq p\}$ has size $<\mu$;
6. For all $p' \leq p$ in P, $q \mapsto w(p, q)$ forms an order-preserving map from $W(p')$ to $W(p)$;
7. Suppose that $U \subseteq P$ is a 0-open set, i.e., $r \in U$ iff $P_0^r \subseteq U$. Then, for all $p \in P$ and $n < \omega$, there is $q \in P_0^p$, such that, either $P_n^q \cap U = \emptyset$ or $P_n^q \subseteq U$.

Comparing iteration theorems: regulars vs successors of singulars

Successors of Regular cardinals (CDMMS)	Successor of Singular cardinals (PRS)
$\kappa^{<\kappa} = \kappa$	$\mathbb{1} \Vdash_{\mathbb{P}}$ "κ singular & $\check{\mu} = \kappa^+$" and $\mu^{<\mu} = \mu$
κ-closedness+countably parallel closed	CPP + layered closedness
κ^+-stationary-cc	μ^+-Linked$_0$-property
well-metness	Not available (e.g. **EBPF**)

The key concept that allows to preserve the above properties (as well as the others definining a Σ-Prikry forcing) along the iteration is the notion of forking projection.

The set-up of forking projections

1. $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is a Σ-Prikry triple with $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$.
2. $(\mathbb{A}, \ell_{\mathbb{A}})$ is a graded poset, $\mathbb{A} := (A, \trianglelefteq)$, joint with a function $c_{\mathbb{A}} : A \to \mathfrak{M}$, where $\mathfrak{M}$ is some canonical structure of size μ.

Definition

We say that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ forking projects onto $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ iff there are maps π and $\pitchfork$ as follows:

1. π is a projection from $\mathbb{A}$ onto $\mathbb{P}$ and $\ell_{\mathbb{A}} = \ell_{\mathbb{P}} \circ \pi$.
2. For each $p \in P$, the set $\{a \in A \mid \pi(a) = p\}$ contains a $\unlhd$-greatest element denoted by $\lceil p \rceil^{\mathbb{A}}$.
3. For each $a \in A$, $\pitchfork(a)$ is a order-preserving map from $\mathbb{P} \downarrow \pi(a)$ to $\mathbb{A} \downarrow a$. Furthermore, $\pitchfork(a) \restriction W(\pi(a))$ is a bijection onto $W(a)$.
4. For all $n, m < \omega$, $b \unlhd^{n+m} a$, $m(a, b) = \pitchfork(a)(m(\pi(a), \pi(b)))$.
5. For all $a \in A$, $\pitchfork(a)$ splits π; i.e., $\pi(\pitchfork(a)(q)) = q$, for $q \leq \pi(a)$.
6. For all $a \in A$ and $q \leq \pi(a)$, $a = \lceil \pi(a) \rceil^{\mathbb{A}}$ iff $\pitchfork(a)(q) = \lceil q \rceil^{\mathbb{A}}$.
7. For all $a \in A$, $a' \unlhd^0 a$ and $r \leq^0 \pi(a')$, $\pitchfork(a')(r) \unlhd^0 \pitchfork(a)(r)$.
8. For all $a, a' \in A$, if $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$ then $c_{\mathbb{P}}(\pi(a)) = c_{\mathbb{P}}(\pi(a'))$ and for all $r \in P_0^{\pi(a)} \cap P_0^{\pi(a')}$, $\pitchfork(a)(r) = \pitchfork(a')(r)$.

Note

In case there are maps π and $\pitchfork$ just satisfying (1)-(7) of the above we will say that $(\mathbb{A}, \ell_{\mathbb{A}})$ forking projects onto $(\mathbb{P}, \ell_{\mathbb{P}})$

Some intuitions

1. Clause (2) states that any condition $p \in P$ "lifts" to a condition in A. The condition $\lceil p \rceil^{\mathbb{A}}$ is analogous to $(p, \dot{\mathbb{1}}_{\mathbb{Q}})$ in a two-step iteration $\mathbb{A} = \mathbb{P} * \dot{\mathbb{Q}}$.
2. Intuitively speaking, $\pitchfork(a)(p)$ give us the $\trianglelefteq$-greatest extension of a whose projection under π is p.
3. Clauses (1)+(3)+(4)+(5) imply that the map defined by $w(\pi(a), \pi(b)) \mapsto \pitchfork(a)(w(\pi(a), \pi(b)))$ establishes an isomorphism between the a-tree $(W(a), \trianglerighteq)$ and the $\pi(a)$-tree $(W(\pi(a)), \geq)$.
   - By (3), $\pitchfork(a)$ is order-preserving.
   - Let $w(a, b_1) \trianglelefteq w(a, b_0)$. By (4), $w(a, b_i) = \pitchfork(a)(w(\pi(a), \pi(b_i))$ and combining (5) and (1) $w(\pi(a), \pi(b_1)) \leq w(\pi(a), \pi(b_0))$.

Some intuitions

1. Clause (6) can be interpreted as follows. A condition $a \in A$ is a lift if and only if $\pitchfork(a)(q)$ is a lift, for each $q \leq \pi(a)$.
2. Let $\Sigma := \langle \kappa_n \mid n < \omega \rangle$. Clause (7) is key to guarantee that for each $n < \omega$, $\mathbb{A}_n$ is κ_n-directed closed.
3. Observe that $\pitchfork(a)(r) = \pitchfork(a')(r) \in A_0^a \cap A_0^{a'}$. Thus, (8) claims that $c_{\mathbb{A}}$ satisfies a strong form of the μ^+-Linkedness$_0$-property: namely, this property is witnessed by any condition of the form $\pitchfork(a)(r)$, for any $r \in P_0^{\pi(a)} \cap P_0^{\pi(a')}$.

Main theorem

Suppose that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of regular uncountable cardinals, converging to a cardinal κ. Let us say that a notion of forcing $\mathbb{P}$ is nice if $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$ and $\mathbb{P} \subseteq H_{\mu^+}$. Now, suppose that:

- $(\mathbb{Q}, \ell, c)$ is a nice Σ-Prikry notion of forcing;
- $\mathbb{A}(\cdot, \cdot)$ is a functor that produces for every nice Σ-Prikry notion of forcing $\mathbb{P}$ and every $\mathbb{P}$-name $z \in H_{\mu^+}$, a corresponding nice Σ-Prikry notion of forcing $(\mathbb{A}(\mathbb{P}, z), \ell', c')$ that admits a forking projection to $\mathbb{P}$ and satisfies some additional properties;
- $\mu^{<\mu} = \mu$ and $2^\mu = \mu^+$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \mu^+ \rangle$ of H_{μ^+}.

Then there exists a $<\mu$-supported sequence $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha) \mid \alpha \leq \mu^+ \rangle$ of nice Σ-Prikry forcings such that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$, $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$ and, for every pair $\alpha \leq \beta < \mu^+$, $(\mathbb{P}_\beta, \ell_\beta, c_\beta)$ forking projects onto $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ and $(\mathbb{P}_{\mu^+}, \ell_{\mu^+})$ forking projects onto $(\mathbb{P}_\beta, \ell_\beta)$.

Let us iterate Σ-Prikry forcings

Let us assume throughout that $\mu^{<\mu} = \mu$.

Building block I

We are given $(\mathbb{Q}, \ell_{\mathbb{Q}}, c_{\mathbb{Q}})$ a Σ-Prikry forcing such that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$, $\mathbb{Q} \subseteq H_{\mu^+}$ and $\mathbb{1} \Vdash_{\mathbb{P}}$ "κ is singular".

Building block II

We are given a function $\psi\colon \mu^+ \to H_{\mu^+}$.

Note

The typical choice of ψ in applications are bookkeping functions: i.e., ψ is such that $|\psi^{-1}\{x\}| = \mu^+$, for each $x \in H_{\mu^+}$. For this one just need to add $|H_{\mu^+}| = \mu^+$ to the above assumptions.

Building block III

For every nice Σ-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$, every $r^\star \in P$, and every $\mathbb{P}$-name $z \in H_{\mu^+}$, we are given a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ such that:

1. $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ as witnessed by maps $\pitchfork$ and π;
2. $\mathbb{A} = (A, \trianglelefteq)$ is nice;
3. **Mixing property**: for all $a \in A$, $m < \omega$, and $p' \leq^0 \pi(a)$, and for every function $g : W_m(p') \to A$ satisfying $g(r) \trianglelefteq a$ and $\pi(g(r)) = r$ for every $r \in W_m(p')$, there exists $b \trianglelefteq^0 a$ with $\pi(b) = p'$ such that $\pitchfork(b)(r) \trianglelefteq^0 g(r)$ for every $r \in W_m(p')$.

By virtue of a lemma concerning canonical forms we may further assume:

1. each element of A is a pair (x, y) with $\pi(x, y) = x$;
2. for every $a \in A$, $\lceil \pi(a) \rceil^{\mathbb{A}} = (\pi(a), \emptyset)$;
3. for every $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}(\lceil p \rceil^{\mathbb{A}}) = c_{\mathbb{A}}(\lceil q \rceil^{\mathbb{A}})$;

$<\mu$-supported, μ^+-iterations of Σ-Prikry forcing

Since $\mu^{<\mu} = \mu$,

- Fix $e_\alpha \colon \alpha \to \mu$ an injection, for each $\alpha < \mu^+$;
- Let $\langle e^i \mid i < \mu \rangle$, $e^i \colon \mu^+ \to \mu$, be such that for each $e : C \to \mu$ with $C \in [\mu^+]^{<\mu}$ there is $i < \mu$ such that $e \subseteq e^i$ (Engelking-Karlowicz).

Notation

1. For the ease of notation, let us write $\emptyset$ rather than $\mathbb{1}_{\mathbb{Q}}$.
2. For each $\alpha \leq \mu^+$, $\emptyset_\alpha := \{\langle \beta, \emptyset \rangle \mid \beta < \alpha\}$.
3. For each $\gamma \leq \alpha \leq \mu^+$, p a γ-sequence and q an α-sequence,

$$p * q := \begin{cases} q(\beta), & \gamma \leq \beta < \alpha; \\ p(\beta), & \text{otherwise.} \end{cases}$$

4. For each sequence p, $B_p := \{\beta + 1 \mid \beta \in \mathrm{dom}(p) \,\&\, p(\beta) \neq \emptyset\}$.

We define our iteration by induction on $\alpha \leq \mu^+$.

1. $\mathbb{P}_0 := (\{\emptyset\}, \leq_0)$ be the trivial forcing.
2. $\mathbb{P}_1 := ({}^{\{\emptyset\}}Q, \leq_1)$ where $p \leq_1 q$ iff $p(0) \leq_{\mathbb{Q}} q(0)$, $\ell_1(p) := \ell_{\mathbb{Q}}(p(0))$ and $c_1(p) := c_{\mathbb{Q}}(p(0))$. Besides, $\pi_{1,0} : P_1 \to \{\emptyset\}$, $\pitchfork_{1,0} : P_1 \to \{\emptyset\}$ and $\pitchfork_{1,1} := \mathrm{id}$.

Successor stage $\alpha + 1$

Suppose $\langle(\mathbb{P}_\beta, \ell_\beta, c_\beta), \langle \pitchfork_{\beta,\gamma}, \pi_{\beta,\gamma} \mid \gamma \leq \beta \leq \alpha\rangle\rangle$ was already defined.

- Suppose that $\psi(\alpha) = (\beta, r, \sigma)$ where $\beta < \alpha, r \in P_\beta$ and σ is a $\mathbb{P}_\beta$-name. Then appeal to Building Block III w.r.t. $r^\star := r * \emptyset_\alpha$, $z := \{(\tau^{\beta,\alpha}, p * \emptyset_\alpha) \mid (\tau, p) \in \sigma)\}$ to get $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ a Σ-Prikry triple joint with two maps π and $\pitchfork$ witnessing that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ forking projects onto $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$.
- Otherwise, appeal to Building Block III w.r.t. $r^\star := \emptyset_\alpha$ and $z := \emptyset$ and get the corresponding Σ-Prikry forcing joint with maps π and $\pitchfork$.

Successor stage $\alpha+1$ (continuation)

Once $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$, π and $\pitchfork$ are obtained, we define $(\mathbb{P}_{\alpha+1}, \ell_{\alpha+1}, c_{\alpha+1})$ and the maps $\langle \pitchfork_{\alpha+1,\beta}, \pi_{\alpha+1,\beta} \mid 1 \leq \beta \leq \alpha+1 \rangle$ as follows:

- $P_{\alpha+1} := \{x^\frown\langle y\rangle \mid (x,y) \in A\}$ and

$$p \leq_{\alpha+1} q \iff (p \restriction \alpha, p(\alpha)) \trianglelefteq (q \restriction \alpha, q(\alpha)).$$

- $\pi_{\alpha+1,\beta}(p) := p \restriction \beta$, $\ell_{\alpha+1} := \ell_1 \circ \pi_{\alpha+1,1}$.
- $c_{\alpha+1}(p) := c_{\mathbb{A}}(p \restriction \alpha, p(\alpha))$.
- $\pitchfork_{\alpha+1,\alpha+1} :=$ id and for each $\beta \leq \alpha$, $p \in P_\alpha$ and $r \in P_\beta$

$$\pitchfork_{\alpha+1,\beta}(p)(r) := x^\frown\langle y\rangle \text{ iff } \pitchfork(p \restriction \alpha, p(\alpha))(\pitchfork_{\alpha,\beta}(p \restriction \alpha)(r)) = (x,y)$$

Limit stage $0 < \alpha \leq \mu^+$

Suppose $\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta), \langle \pitchfork_{\beta,\gamma}, \pi_{\beta,\gamma} \mid \gamma \leq \beta < \alpha \rangle \rangle$ was already defined. We define $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ and the maps $\langle \pitchfork_{\alpha,\beta}, \pi_{\alpha,\beta} \mid 1 \leq \beta \leq \alpha \rangle$ as follows:

- Let P_α be the set of all α-sequences p such that, for each $\beta < \alpha$, $p \restriction \beta \in P_\beta$ and $|B_p| < \mu$. Define $p \leq_\alpha q$ in the natural way.
- $\pi_{\alpha,\beta}(p) := p \restriction \beta$, $\ell_\alpha := \ell_1 \circ \pi_{\alpha,1}$.
- $\pitchfork_{\alpha,\alpha} :=$ id and for each $\beta < \alpha$, $p \in P_\alpha$ and $r \in P_\beta$,

$$\pitchfork_{\alpha,\beta}(p)(r) := \bigcup_{\beta \leq \delta < \alpha} \pitchfork_{\delta,\beta}(p \restriction \delta)(r).$$

Idea: Guarantee that $\pitchfork_{\alpha,\beta}(p)(r) \restriction \delta = \pitchfork_{\delta,\gamma}(p \restriction \delta)(r)$ as we want to preserve the existence of forking projections.

Limit stage $0 < \alpha \leq \mu^+$ (continuation)

For the definition of c_α we distinguish two cases: either $\alpha < \mu^+$ or $\alpha = \mu^+$.

- If $\alpha < \mu^+$, define $c_\alpha(p) := \{(e_\alpha(\gamma), c_\gamma(p \restriction \gamma)) \mid \gamma \in B_p\}$.
- Otherwise, for each $p \in P_{\mu^+}$ set $C := \mathrm{cl}(B_p)$ and, for each $\gamma \in C$, set

$$f_p(\gamma) := (e_\gamma[C \cap \gamma], c_\gamma(p \restriction \gamma)).$$

  Finally, define $c_{\mu^+}(p) := \min\{i < \mu \mid f_p \subseteq e^i\}$.

The idea when $0 < \alpha < \mu^+$ is limit

We want to devise c_α in such a way that Clause (8) of forking projections is true for each $1 \leq \gamma \leq \alpha$. In particular this will show that c_α witnesses the μ^+-Linked$_0$-property of $\mathbb{P}_\alpha$.

Observe that if $c_\alpha(p) = c_\alpha(q)$ then $B := B_p = B_q$ and

$$(\star) \quad c_\gamma(p \restriction \gamma) = c_\gamma(q \restriction \gamma), \text{ for each } \gamma \in B.$$

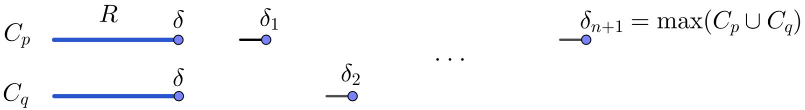
The moral is that, if we have forking projections between all the stages $\beta \leq \gamma < \alpha$, the coordinates $\gamma \in \alpha \setminus B$ are "not important", i.e. $(\star)$ yields $c_\gamma(p \restriction \gamma) = c_\gamma(q \restriction \gamma)$, for each $\gamma \leq \alpha$.

Once this is proved, it is not hard to check that $\pitchfork_{\alpha,\gamma}(p)(r) = \pitchfork_{\alpha,\gamma}(q)(r)$, for each $r \in (P_\gamma)_0^{p\restriction\gamma} \cap (P_\gamma)_0^{q\restriction\gamma}$.

Sketch: c_{μ^+} witnesses the μ^+-Linked$_0$-property

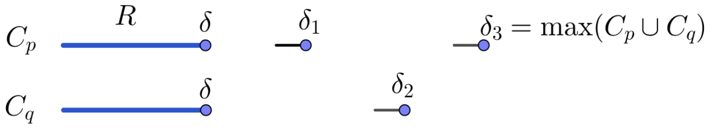
The caveat now is that there are no forking projections between $(\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})$ and $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$, for $\alpha < \mu^+$. We will be assuming that $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ is Σ-Prikry, for each $\alpha < \mu^+$.

We devise c_{μ^+} in such a way that, if $c_{\mu^+}(p) = c_{\mu^+}(q)$ and $C := \mathrm{cl}(B_p)$ and $C_q := \mathrm{cl}(B_q)$, then both C_p and C_q can be represented as follows:



Let us now show how to use this to define $r \in (P_{\mu^+})^p_0 \cap (P_{\mu^+})^q_0$.

Set $C := \mathrm{cl}(B_p)$ and $C_q := \mathrm{cl}(B_q)$ and assume $c_{\mu^+}(p) = c_{\mu^+}(q)$. For simplicity, say $n = 2$.



- Since $c_{\mu^+}(p) = c_{\mu^+}(q)$ entails $f_p \restriction R = f_q \restriction R$, and $\delta \in R$, it follows that $f_p(\delta) = f_q(\delta)$. In particular, $c_\delta(p \restriction \delta) = c_\delta(q \restriction \delta)$. Thus, there is $r \in (P_\delta)_0^{p\restriction\delta} \cap (P_\delta)_0^{q\restriction\delta}$. Set $r_0 := r$.
- Now, we begin "copying" the information:
  - $r_1 := \pitchfork_{\delta_1,\delta_0}(p \restriction \delta_1)(r_0)$, $r_2 := \pitchfork_{\delta_2,\delta_1}(q \restriction \delta_2)(r_1)$ and $r_3 := \pitchfork_{\delta_3,\delta_2}(p \restriction \delta_3)(r_2)$.
  - $r^\star := r_3 * \emptyset_{\mu^+}$.

By construction it is not hard to check that $r^\star \in (P_{\mu^+})_0^p \cap (P_{\mu^+})_0^q$.

Sketch: for each $1 \leq \alpha \leq \mu^+$, $\mathbb{P}_\alpha$ has the CPP

To enlighten the presentation let us prove the result for a graded poset $(\mathbb{A}, \ell_{\mathbb{A}})$ which forking projects onto $(\mathbb{P}, \ell_{\mathbb{P}})$ and $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is Σ-Prikry. Denote by π and $\pitchfork$ the corresponding maps witnessing this.

The main two ingredients are:

1. Mixing lemma.
2. CPP of $\mathbb{P}$.

Sketch of proof

Let $a \in A$ and $D \subseteq A$ be a 0-open set. We want to find $b \trianglelefteq^0 a$ and $n < \omega$ such that either $A^b_n \subseteq D$ or $A^b_n \cap D = \emptyset$. Set $D_a := D \downarrow a$, $U := \pi[D_a]$ and $p := \pi(a)$.
Using elementary properties of $\pitchfork$ one can show that U is a 0-open set in P. Thus, by the CPP for P, there is $q \leq^0 p$ and $n < \omega$ such that, either $P^q_n \subseteq U$, or $P^q_n \cap U = \emptyset$.

Sketch of proof (continuation)

▶▶ $P_n^q \cap U = \emptyset$: Set $b := \pitchfork(a)(q)$. It is routine to check that $A_n^b \cap D = \emptyset$, so we are done.
▶▶ $P_n^q \subseteq U$: Let $g : W_n(q) \to D_a$ be such that $\pi(g(r)) = r$. Now use the mixing lemma to find $b \trianglelefteq^0 a$ with $\pi(b) = q$ such that $\pitchfork(b)(r) \trianglelefteq^0 g(r)$. By 0-openes of D_a, $\pitchfork(b)[W_n(q)] \subseteq D_a$ and this is the same as $W_n(b) \subseteq D_a$. Again, by the 0-openess of D_a, $A_n^b \subseteq D$, as desired.

The papers

1. **Sigma-Prikry forcing I: The axioms**, Submitted to Canadian Journal of Mathematics (2019).
2. **Sigma-Prikry forcing II: Iteration Scheme**, Submitted to Journal of Mathematical Logic (2019).

Find the papers and the slides of Lecture #1 here!
http://assafrinot.com/t/sigma-prikry