Σ-Prikry forcings and their iterations

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Joint work with A. Rinot & D. Sinapova

1. **Sigma-Prikry forcing I: The axioms**, Submitted to Canadian Journal of Mathematics (2019).
2. **Sigma-Prikry forcing II: Iteration Scheme**, Submitted to Journal of Mathematical Logic (2019).

The subject matter of this talk is **Prikry-type forcings**

- **Main role:** Generally devised to **change cofinalities** and blow up the power set of a singular cardinal
  - Due to foundational reasons this needs Very Large Cardinals (Jensen)
- Have found several connections/applications in central areas of Set Theory
  - The Singular Cardinals Problem (Prikry, Magidor, Gitik...)
  - Identity crises phenomena (Magidor, Apter...)
  - Inner Model Theory (Mitchell, Cummings & Schimerling...)

Motivating goal

Theorem

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a cofinality-preserving extension where

1. $\kappa = \sup_{n<\omega} \kappa_n$ is a strong limit cardinal;
2. $\neg\mathrm{SCH}_\kappa$;
3. $\mathrm{Refl}({<}\omega, \kappa^+)$ holds.

- Around the same time, it was also proved by Ben-Neria-Hayut-Unger and soon after by Gitik.
- Their proof avoids iterated forcing and extends to uncountable cofinality. The novelty in our approach is the iteration scheme for Σ-Prikry forcings.
- Announced by A. Sharon in 2005.

Prikry-type forcings

The first representative of this family is the so-called **Prikry forcing**:

- Let κ be a measurable cardinal.
- Let $\mathcal{U}$ be a non-principal, normal and κ-complete ultrafilter over it (**measure**).

Definition (Prikry 1970)

Prikry forcing $\mathbb{P}_{\mathcal{U}}$ is the poset whose conditions are pairs (s, A) where

1. $s \in [\kappa]^{<\omega}$ strictly increasing;
2. $A \in \mathcal{U}$ with $\max(s) < \min(A)$.

We will write $(s, A) \leq (t, B)$ iff s end-extends t, $s \setminus t \subseteq B$ and $A \subseteq B$.
We consider an additional ordering $\leq^* \subseteq \leq$ defined as $(s, A) \leq^* (t, B)$ iff $(s, A) \leq (t, B)$ and $s = t$.

- For each $n < \omega$, let $\mathbb{P}_n$ be the subposet of $\mathbb{P}$ whose conditions (s, A) have $|s| = n$ joint with the trivial condition $\mathbb{1}$.

Properties of $\mathbb{P}$

1. $\mathbb{P}$ is κ-centered, hence cardinals $\geq \kappa^+$ are preserved;
2. $\mathbb{P}$ forces $\mathrm{cof}(\kappa) = \omega$.
3. $\mathbb{P}$ does not add bounded subsets to κ. In particular, cardinals $\leq \kappa$ are preserved.

(1) and (2) of above are easy to prove but (3) is not so immediate:

1. for each $n < \omega$, $(\mathbb{P}_n, \leq)$ is κ-closed;
2. $\mathbb{P}$ satisfies the Prikry property.

Prikry property

For each $p \in \mathbb{P}$ and each sentence φ in the language of forcing, there is $q \leq^* p$ such that q decides φ.

In other words, the set $D_\varphi = \{p \in \mathbb{P} \mid p \parallel \varphi\}$ is $\leq^*$-dense.

Lemma (Prikry)

Prikry forcing has the Prikry property.

Theorem (Prikry)

If there is a measurable cardinal then there is a cardinal-preserving generic extension where the measurable becomes a singular strong limit cardinal of countable cofinality.

Some Examples

1. Prikry forcing (Prikry).
2. Supercompact Prikry forcing (Magidor).
3. Gitik-Sharon forcing.
4. Magidor forcing.
5. Radin forcing (Radin & Woodin)
6. Diagonal Supercompact Magidor forcing (Sinapova)
7. Extender Based Prikry forcing (EBPF) (Gitik & Magidor)
8. Extender Based Radin forcing (Merimovich)

The aim of our project

Our project has two goals:

1. Provide an abstract framework which allows a systematic study of Prikry-type forcings
2. Devise a viable iteration scheme for these forcings

The Σ-Prikry framework

What characterize Prikry-type posets?

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2. For all length n, $\mathbb{P}_n := \{p \mid \ell(p) = n\}$ is *"closed enough"* (e.g. in Prikry forcing $\mathbb{P}_n$ is κ-closed)

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4. Decision by pure extensions (e.g. the Prikry property).

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We want to be able to iterate, so we will in addition require a quite prevalent feature

(*) $\mathbb{P}$ has some good chain condition

Σ-Prikry forcings

Definition (Graded poset)

We say that $(\mathbb{P}, \ell)$ is a graded poset if $\mathbb{P} = (P, \leq)$ is a poset, $\ell : P \to \omega$ is a surjection, and, for all $p \in P$, the following are true:

- For every $q \leq p$, $\ell(q) \geq \ell(p)$;
- There exists $q \leq p$ with $\ell(q) = \ell(p) + 1$.

Notation

For a graded poset as above we write

1. $P_n := \{p \in P \mid \ell(p) = n\}$.
2. $P_n^p := \{q \in P \mid q \leq p \,\&\, \ell(q) = \ell(p) + n\}$.

For ease of notation we sometimes write $q \leq^n p$ rather than $q \in P_n^p$.

The Σ-Prikry framework

1. $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}$;
2. $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals with $\kappa := \sup_{n<\omega} \kappa_n$;
3. μ is a cardinal such that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$;
4. $\ell : P \to \omega$ and $c : P \to \mu$ are functions;

Definition (Σ-Prikry forcing)

We say that $(\mathbb{P}, \ell, c)$ is Σ-Prikry iff all of the following hold:

1. $(\mathbb{P}, \ell)$ is a graded poset;
2. For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$ is κ_n-directed-closed;
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P_0^p \cap P_0^q$ is non-empty;
4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{r \leq^n p \mid q \leq^m r\}$ contains a $\leq$-largest condition $m(p, q)$. In the particular case that $m = 0$, we write $w(p, q)$ instead of $m(p, q)$;

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5. For all $p \in P$, the set $W(p) := \{w(p, q) \mid q \leq p\}$ has size $<\mu$;
6. For all $p' \leq p$ in P, $q \mapsto w(p, q)$ forms an order-preserving map from $W(p')$ to $W(p)$;
7. Suppose that $U \subseteq P$ is a 0-open set, i.e., $r \in U$ iff $P_0^r \subseteq U$. Then, for all $p \in P$ and $n < \omega$, there is $q \in P_0^p$, such that, either $P_n^q \cap U = \emptyset$ or $P_n^q \subseteq U$.

Some clarifications

How $m(p,q)$ and $w(p,q)$ look like?

For simplicity let us assume that $\mathbb{P}$ is Prikry forcing. Say $p = (s, A)$ and $q = (s^\frown\langle\alpha,\beta\rangle, B)$. Let $m \leq \ell(q) - \ell(p)$.

- Intuitively, $w(p,q)$ is the $\leq$-greatest interpolation between p and q with length $\ell(q)$. In this case, $w(p,q) = (s^\frown\langle\alpha,\beta\rangle, A \setminus \beta + 1)$.
- In general, $m(p,q)$ is the $\leq$-greatest interpolation between p and q with length $\ell(q) - m$. In this case, $1(p,q) = (s^\frown\langle\alpha\rangle, A \setminus \alpha + 1)$ and $2(p,q) = (s, A) = p$.

Convention

For each $n < \omega$ and $p \in P$, we write $W_n(p) := \{w(p,q) \mid q \leq^n p\}$. Hence, $W(p) = \bigcup_{n<\omega} W_n(p)$.

Novelties of the Σ-Prikry framework

μ^+-Linked$_0$-property

For all $p, q \in P$, if $c(p) = c(q)$, then $P_0^p \cap P_0^q$ is non-empty.

Complete Prikry Property

Suppose that $U \subseteq P$ is a 0-open set, i.e., $r \in U$ iff $P_0^r \subseteq U$. Then, for all $p \in P$ and $n < \omega$, there is $q \in P_0^p$, such that, either $P_n^q \cap U = \emptyset$ or $P_n^q \subseteq U$.

- The first one is a strong form of μ^+-2-Linkedness, hence a strengthening of the μ^+-cc.
- The second one is inspired by the Complete Ramsey Property. Captures two features of Prikry-type forcings: the **Prikry Property** and the **Strong Prikry Property** (see next slide)
- Both are crucial to define viable iterations of Σ-Prikry forcings

CPP yiels the SPP and the PP

Proposition

Let $\mathbb{P}$ be some Σ-Prikry forcing. Then the following are true:

1. $\mathbb{P}$ has the Prikry property.
2. $\mathbb{P}$ has the Strong Prikry property; namely, for each dense open set $D \subseteq P$ and each $p \in P$, there is $q \leq^0 p$ and $n < \omega$ such that $P^q_m \subseteq D$, for each $m \geq n$.

For the proof we use the key concept of 0-open coloring:

Definition

Let $(\mathbb{P}, \ell, c)$ be a Σ-Prikry triple. A 0-open coloring $d : P \to \theta$ is a map such that for each pair $p' \leq^0 p$ of conditions in P, $d(p) \in \{0, d(p')\}$.
We say that $H \subseteq P$ is a set of indiscernibles for d if for each $p, q \in H$, $d(p) = d(q)$, provided $\ell(p) = \ell(q)$.

CPP yiels the SPP and the PP

Lemma

Let $(\mathbb{P}, \ell, c)$ be a Σ-Prikry triple. For each $p \in P$, $n \geq 2$ and each 0-open coloring $d : P \to n$, there is $q \leq^0 p$ such that the set of conditions of P below q is a set of indiscernibles for d.

The CPP yields the PP

Let $p \in P$ and φ a sentence in the language of forcing. Define $d : P \to 3$ as

$$d(r) := \begin{cases} 1, & \text{if } r \Vdash_{\mathbb{P}} \varphi; \\ 2, & \text{if } r \Vdash_{\mathbb{P}} \neg\varphi; \\ 0, & \text{otherwise.} \end{cases}$$

Appeal to the above lemma to find $q \leq^0 p$ such that P^q is a set of indescirnibles for d. It is not hard to check that q already decides φ.

CPP yiels the SPP and the PP

Lemma

Let $(\mathbb{P}, \ell, c)$ be a Σ-Prikry triple. For each $p \in P$, $n \geq 2$ and each 0-open coloring $d : P \to n$, there is $q \leq^0 p$ such that the set of conditions of P below q is a set of indiscernibles for d.

The CPP yields the SPP

Let $p \in P$ and D be an open dense set. Define $d : P \to 2$ as $d(r) := 1$ iff $r \in D$. Appealing to the lemma we get $q \leq^0 p$ such that P^q is a set of indescirnibles for d. Since D is dense, there is $n < \omega$ and $r \leq^n q$ such that $r \in D$. By definition of d, $d(r) = 1$, hence $P^q_n \subseteq D$. Finally the opennes of D yields the desired result; that is, $P^q_m \subseteq D$, for each $m \geq n$.

Other properties of Σ-Prikry forcings

Proposition

Let $\mathbb{P} := (P, \leq)$ be some Σ-Prikry forcing and $p \in P$. Then, the following are true:

1. $\mathbb{P}$ does not add bounded subsets of κ;
2. For each $\nu \geq \kappa$ regular, and each $p \in P$, if $p \Vdash_{\mathbb{P}} \mathrm{cof}(\nu) < \kappa$ then there is $p' \leq p$ such that $|W(p')| \geq \nu$.
3. Assume $\mathbb{1} \Vdash_{\mathbb{P}}$ "κ is singular". Then, $\mu = \kappa^+$ iff $|W(p)| \leq \kappa$, for each $p \in P$.
4. For each $n < \omega$, $W_n(p)$ is a maximal antichain below p.
5. Any two compatible elements of $W(p)$ are comparable. Thus, $(W(p), \geq)$ is a tree **(the p-tree)**
6. $c \restriction W(p)$ is injective.

Some examples: Prikry forcing

Definition (Prikry 1970)

Prikry forcing $\mathbb{P}$ is the poset whose conditions are pairs (s, A) where

1. $s \in [\kappa]^{<\omega}$ strictly increasing;
2. $A \in \mathcal{U}$ with $\max(s) < \min(A)$.

$(s, A) \leq (t, B)$ iff s end-extends t, $s \setminus t \subseteq B$ and $A \subseteq B$.

Prikry forcing is Σ-Prikry

1. Σ is the constant ω-sequence with value κ and $\mu = \kappa^+$;
2. $\ell(s, A) := |s|$;
3. $c(s, A) := s$;

Some examples: Gitik-Sharon forcing

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals. Suppose that $\mathcal{U}$ is a supercompact measure on $\mathcal{P}_{\kappa_0}(\mu^+)$, and let $\mathcal{U}_n$ be its projection onto $\mathcal{P}_{\kappa_0}(\kappa_n)$.

Definition (Gitik & Sharon 2008)

Conditions in $\mathbb{P}$ are sequences $p = \langle x_0^p, \dots, x_{n-1}^p A_n^p, A_{n+1}^p, \dots \rangle$ such that the following holds:

1. $x_i \in \mathcal{P}_{\kappa_0}(\kappa_i)$.
2. $x_i \prec x_{i+1}$ (i.e. $\mathrm{otp}(x_i) < \mathrm{otp}(x_{i+1} \cap \kappa_0)$).
3. $A_k \in \mathcal{U}_k$ and $\{x \in A_k \mid x_{n-1}^p \prec x\} \subseteq A_k$.

The order is the usual: we extend the stems by picking elements from the measure one sets, and then shrink the measure one sets.

Some examples: Gitik-Sharon forcing

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GS-poset is Σ-Prikry

1. Σ is the constant ω-sequence with value κ_0 and $\mu = (\sup_{n<\omega} \kappa_n)^+$.
2. $\ell(p) := |\langle x_0^p, \dots, x_{n-1}^p \rangle|$.
3. $c(p) := \langle x_0^p, \dots, x_{n-1} \rangle$.

Some examples: The Extender-Based Prikry forcing

The set-up

- $\langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence of cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := 2^\mu$;
- $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$;
- for each $n < \omega$, κ_n carries a $(\kappa_n, \lambda + 1)$-extender E_n.

In particular, for each $n < \omega$, we are assuming that there is an elementary embedding $j_n : V \to M_n$with $\mathrm{crit}(j) = \kappa_n$ such that M_n is a transitive class, ${}^{\kappa_n}M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda$.

Definition

For each $n < \omega$, and each $\alpha < \lambda$, define $E_{n,\alpha} := \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}$. For each $\alpha, \beta < \lambda$ write $\alpha \leq_{E_n} \beta$ iff $\alpha \leq \beta$ and there is $\pi_{\beta,\alpha} \colon \kappa_n \to \kappa_n$ such that $j_n(\pi_{\beta,\alpha})(\beta) = \alpha$.

Definition

For $n < \omega$, $\mathbb{Q}_{n0}$ is defined as follows:

$(0)_n$ $\mathbb{Q}_{n0} := (Q_{n0}, \leq_{n0})$, where elements of Q_{n0} are triples $p = (a^p, A^p, f^p)$ meeting the following requirements:

1. f^p is a function from some $x \in [\lambda]^{\leq\kappa}$ to κ_n;
2. $a^p \in [\lambda]^{<\kappa_n}$, and a^p contains a $\leq_{E_n}$-maximal element, which hereafter is denoted by $\mathrm{mc}(a^p)$;
3. $\mathrm{dom}(f^p) \cap a^p = \emptyset$;
4. $A^p \in E_{n,\mathrm{mc}(a^p)}$;
5. if $\beta < \alpha$ is a pair in a, for all $\nu \in A$, $\pi_{\mathrm{mc}(a^p)\beta}(\nu) < \pi_{\mathrm{mc}(a^p)\alpha}(\nu)$;
6. if $\alpha, \beta, \gamma \in a$ with $\gamma \leq_{E_n} \beta \leq_{E_n} \alpha$, then, for all $\nu \in \pi_{\mathrm{mc}(a^p)\alpha}``A$, $\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))$.

The ordering $\leq_{n0}$ is defined as follows: $(a^p, A^p, f^p) \leq_{n0} (b^q, B^q, g^q)$ iff the following are satisfied:

(i) $f^p \supseteq g^q$,
(ii) $a^p \supseteq b^q$,
(iii) $\pi_{\mathrm{mc}(a^p)\mathrm{mc}(b^q)}``A^p \subseteq B^q$.

Definition

For $n < \omega$, $\mathbb{Q}_{n1}$ is defined as follows:

$(1)_n$ $\mathbb{Q}_{n1} := (Q_{n1}, \leq_{n1})$, where $Q_{n1} := \bigcup\{{}^x\kappa_n \mid x \in [\lambda]^{\leq\kappa}\}$ and $\leq_{n1} := \supseteq$.

Essentially $\mathbb{Q}_{n1}$ is Cohen forcing $\mathrm{Add}(\kappa^+, \lambda)$.

Definition

For $n < \omega$, $\mathbb{Q}_n$ is defined as

$(2)_n$ $\mathbb{Q}_n := (Q_{n0} \cup Q_{n1}, \leq_n)$.

The ordering $\leq_n$ is defined as follows: for each $p, q \in Q_n$, $p \leq_n q$ iff

1. either $p, q \in Q_{ni}$ for some $i \in 2$ and $p \leq_{ni} q$, or
2. $p \in Q_{n1}$, $q \in Q_{n0}$ and, for some $\nu \in A$, $p \leq_{n1} q^\frown\langle\nu\rangle$, where

$$q^\frown\langle\nu\rangle := f^q \cup \{(\beta, \pi_{\mathrm{mc}(a^q),\beta}(\nu)) \mid \beta \in a^q\}.$$

Some examples: The Extender-Based Prikry forcing

Definition

The Extender Based Prikry Forcing is the poset $\mathbb{P} := (P, \leq)$ defined by the following clauses:

- Conditions in P are sequences $p = \langle p_n \mid n < \omega \rangle \in \prod_{n<\omega} Q_n$.
- For all $p, q \in P$, $p \leq q$ iff $p_n \leq_n q_n$ for every $n < \omega$.
- For all $p \in P$:
  - There is $n < \omega$ such that $p_n \in Q_{n0}$;
  - For every $n < \omega$, if $p_n \in Q_{n0}$, then $p_{n+1} \in Q_{n0}$ and $a^{p_n} \subseteq a^{p_{n+1}}$.

The Extender-Based Prikry forcing is Σ-Prikry

1. $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ and $\mu := (\sup_n \kappa_n)^+$.
2. $\ell(p) := \min\{n < \omega \mid p_n \in Q_{n0}\}$.
3. c is more elaborated than in the previous cases.

The function c for the EBPF

Since we are assuming $\mu^\kappa = \mu$ and $2^\mu = \lambda$, let us fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from λ to μ with the property that, for every function $e : x \to \mu$ with $x \in [\lambda]^{\leq \kappa}$, there exists $i < \mu$ with $e \subseteq e^i$.

Definition

For every $f \in \bigcup_{n<\omega} Q_{n1}$, let $i(f) := \min\{i < \mu \mid f \subseteq e^i\}$.
For every $p = (a, A, f) \in \bigcup_{n<\omega} Q_{n0}$, let $i(p)$ be the least $i < \mu$ such that:

- for all $\alpha \in a$, $e^i(\alpha) = 0$;
- for all $\alpha \in \mathrm{dom}(f)$, $e^i(\alpha) = f(\alpha) + 1$.

Finally, for every condition $p = \langle p_n \mid n < \omega \rangle$ in P, let

$$c(p) := \ell(p)^\frown \langle i(p_n) \mid n < \omega \rangle.$$

The function c for the EBPF

The Extender Based Prikry forcing has the μ^+-Linked$_0$-property

Let p, q be two conditions in the EBPF with $c(p) = c(q)$. The goal is to show that p and q are compatible as witnessed by a 0-extension of both conditions. More precisely, we want to prove $P_0^p \cap P_0^q \neq \emptyset$.

Set i be this common value of the c function. By definition, p and q have the same length, say ℓ. Now let $n \geq \ell$. To prove $P_0^p \cap P_0^q \neq \emptyset$ it suffices to check that $a_n^p \cap \mathrm{dom}(f_n^q) = a_n^q \cap \mathrm{dom}(f_n^p) = \emptyset$. Let us just check that $a_n^p \cap \mathrm{dom}(f_n^q) = \emptyset$ as the other equality can be proved similarly.

Indeed, since $c(p) = i$ it follows that $e^i \restriction a_n^p = 0$. On the other hand, as $c(q) = i$, $e^i \restriction \mathrm{dom}(f_n^q) \neq 0$. Both equalities combined finally yield $a_n^p \cap \mathrm{dom}(f_n^q) = \emptyset$, as desired.

More examples

1. Supercompact Prikry forcing (Magidor);
2. AIM forcing (Cummings et al.);

Other candidates to be Σ-Prikry

1. Tree Prikry forcing;
2. Strongly Compact Gitik-Sharon forcing;
3. Extender Based Prikry forcing with a single extender;

An interlude on iterations of forcing

Some iteration theorems

(I) The $<\aleph_0$-support iteration of ccc forcing is also ccc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_0}}(ccc) = MA$ **(Solovay-Tennembaum)**

Observation

The above result does not extend to larger supports. Namely, even under the CH, there are countable support iterations of $\aleph_2$-cc + $\aleph_1$-closed forcing which are not $\aleph_2$-cc (Mitchell).

(I) The $<\aleph_0$-support iteration of ccc forcing is also ccc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_0}}(ccc) = MA$ **(Solovay-Tennembaum)**.

(II) Let Γ be the family of well-met, $\aleph_1$-linked and $\aleph_1$-closed forcings. Under the CH, the $<\aleph_1$-support iteration of forcings in Γ is $\aleph_2$-cc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_1}}(\Gamma) := BA$ **(Baumgartner)**

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(III) Let Γ be the family of well-met, $\aleph_2$-stationary-cc and $\aleph_1$-closed forcings with exact upper bounds. Under the CH, the $<\aleph_1$-support iteration of members of Γ is $\aleph_2$-stationary-cc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_1}}(\Gamma)$ **(Shelah)**

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(II) Let Γ be the family of well-met, $\aleph_1$-linked and $\aleph_1$-closed forcings. Under the CH, the $<\aleph_1$-support iteration of forcings in Γ is $\aleph_2$-cc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_1}}(\Gamma) := BA$ **(Baumgartner)**

(III) Let Γ be the family of well-met, $\aleph_2$-stationary-cc and $\aleph_1$-closed forcings with exact upper bounds. Under the CH, the $<\aleph_1$-support iteration of members of Γ is $\aleph_2$-stationary-cc $\Rightarrow$ **Consistency of** $\mathrm{FA}_{2^{\aleph_1}}(\Gamma)$ **(Shelah)**

(IV) Let Γ be the family of well-met, κ^+-stationary-cc, κ-closed and countably parallel closed forcing. Under $\kappa^{<\kappa} = \kappa$, the iteration of $<\kappa$-supported iteration of members of Γ is κ^+-stationary-cc **Consistency of** $\mathrm{FA}_{2^\kappa}(\Gamma)$ **(Cummings et. al)**

Goal

Solve problems about singular cardinals and their successors.

Strategy

Find an analogous iteration theorem for κ being a successor of a singular cardinal.

To be continued in the next lecture