ON BOREL SUBSETS OF GENERALIZED BAIRE SPACES

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Abstract. We develop descriptive set theory in Generalized Baire Spaces without assuming $\kappa^{<\kappa} = \kappa$. We point out that without this assumption the basic topological concepts of Generalized Baire Spaces have to be slightly modified in order to obtain a meaningful theory. This modification has no effect if $\kappa^{<\kappa} = \kappa$. After developing the basic theory we apply it to the question whether the orbits of models of a fixed cardinality κ in the space 2^κ (or κ^κ) are κ-Borel in our generalized sense. It turns out that this question depends, as is the case when $\kappa^{<\kappa} = \kappa$, on stability theoretic properties (structure vs. non-structure) of the first order theory of the model.

1. Introduction

We develop descriptive set theory in generalized Baire spaces without cardinality arithmetic assumptions, such as the Continuum Hypothesis, or the assumption $\kappa^{<\kappa} = \kappa$ for the relevant κ. We immediately observe that in order to develop descriptive set theory in a meaningful way, we have to focus on κ-open sets, i.e. unions of at most κ basic open sets, for the relevant κ, rather than all open sets. There may be simply too many open sets. Of course this makes no difference is we assume $\kappa^{<\kappa} = \kappa$. Moreover, we observe that in order for the concept of κ-open to cover some sets important for our applications, we have to modify slightly the received concept of basic open set. Again, the modification makes no difference if $\kappa^{<\kappa} = \kappa$.

Our applications come from model theory. Indeed, model theory is natural framework for descriptive set theory in generalized Baire spaces. We cannot rely on natural sciences as a source of inspiration in the same way that in classical Baire spaces ω^ω. The idea of a measurement with an ω_1-sequence of successive improvements does not arise in natural science in any apparent way. But uncountable models are essential in the study of first order, as well as infinitary theories.

It is natural to ask e.g. whether the orbit (i.e. isomorphism type) of a model is in some sense Borel, rather than merely analytic. Such investigation has been already done under cardinality arithmetic assumptions ([5], [2],[4], [7], [9]). The purpose of this paper is to prove these results without extra assumptions.

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2. Basic topology

Suppose κ is a regular cardinal. In generalized descriptive set theory the sets κ^κ and 2^κ are usually ([12, 8, 2]) endowed with the topology generated by the basic open sets

$$N_\eta = \{\zeta \in \kappa^\kappa \mid \eta \subseteq \zeta\}, \tag{1}$$

where $\eta : \alpha \to \kappa$ (or $\zeta : \kappa \to \{0,1\}$ and $\eta : \alpha \to \{0,1\}$ in the case of 2^κ) and $\alpha < \kappa$. A set U is then defined as open if it is the union of a family of basic open sets:

$$U = \bigcup_{i \in I} N_{\eta_i}. \tag{2}$$

This definition raises the question, how many basic open sets are there, especially as in the classical Baire space ω^ω and Cantor space 2^ω there are only countably many basic open sets. Obviously, the number of basic open sets (1) is $\kappa^{<\kappa}$. This suggests that the descriptive set theory of spaces such as κ^κ and 2^κ is different according to whether $\kappa^{<\kappa} = \kappa$, $\kappa^{<\kappa} = \kappa^+$, or in general, $\kappa^{<\kappa} > \kappa$.

The spaces built on (1) and (2) have given rise to a lot of research, starting with [12] and [8] among the early ones and e.g. [2] among the more recent ones. In this pioneering research the assumption $\kappa^{<\kappa} = \kappa$ is made virtually without exception. If this assumption is dropped and we allow also the case $\kappa^{<\kappa} \neq \kappa$, a problem with the definition (2) arises:

Lemma 2.1. *Suppose $2^\lambda = 2^\kappa$, $\kappa > \lambda$. Then for all $A \subseteq \kappa^\kappa$ there is a closed $C \subseteq \kappa^\kappa \times 2^\kappa$ such that A is the first projection of C i.e.*

$$A = \{f \in \kappa^\kappa : \exists g \in 2^\kappa((f,g) \in C)\}.$$

Proof. Let $\pi : \kappa^\kappa \to 2^\lambda$ be a bijection and

$$C = \{(\eta,\xi) \in \kappa^\kappa \times 2^\kappa : \eta = \pi^{-1}(\xi \restriction \lambda) \in A\}.$$

Clearly, the first projection of C is A. To prove that C is closed, suppose $(\eta,\xi) \notin C$. If it happens that $\pi^{-1}(\xi \restriction \lambda) \notin A$, then we let

$$U = \{(\eta',\xi') : \xi' \restriction \lambda = \xi \restriction \lambda, \eta' \in \kappa^\kappa\}.$$

The set U is open, $(\eta,\xi) \in U$ and $C \cap U = \emptyset$. On the other hand, if it happens that $\pi^{-1}(\xi \restriction \lambda) \in A$, then there is $\alpha < \kappa$ such that $\eta(\alpha) \neq \pi^{-1}(\xi \restriction \lambda)(\alpha)$ and we let

$$U = \{(\eta',\xi') : \xi' \restriction \lambda = \xi \restriction \lambda, \eta'(\alpha) = \eta(\alpha)\}.$$

The set U is again open, $(\eta,\xi) \in U$ and $C \cap U = \emptyset$.

$\square$

This lemma shows that we cannot hope for any meaningful theory of analytic sets in the case that $2^\omega = 2^\kappa$, $\kappa > \omega$. Still e.g. orbits of structures of size κ are canonical analytic sets in these spaces. A natural way out of the *cul-de-sac* of Lemma 2.1 is to define a set to be "κ-open" if (2) is satisfied with $|I| \le \kappa$. Indeed, this kills the proof of Lemma 2.1 in the sense that the set C therein is not κ-closed, i.e. the complement of a κ-open set. However, it turns out that the concept of κ-open is too restrictive, as the following example shows:

Example 2.2. *Suppose $2^\lambda > \kappa > \lambda$. The set $\{f \in \kappa^\kappa : f(\lambda) = f(\lambda+1)\}$ is clopen but not a union of $\le \kappa$ sets of the form (1).*

This example shows that if we define a set A to be "κ-open" if (2) is satisfies with $|I| \le \kappa$, the extremely simple and "basic" set of Example 2.2, need not be κ-open. Still such sets are the basis of definability theory in infinitary model theory. This leads us to redefine the topology of κ^κ (and 2^κ) in a way which is more natural from the point of view of infinitary model theory, even if $\kappa^{<\kappa} > \kappa$. The new topology coincides with the "old" topology of [2] and [8] in the case that $\kappa^{<\kappa} = \kappa$.

To make a judgement whether some concept is natural or not it is reasonable to use relevant applications as a criterion. Of course there is also an aesthetic aspect. As to applications, we look at infinitary model theory for inspiration and this leads us to make the below definition. However, the definition is aesthetically appealing as well, by virtue of its simplicity.

Definition 2.3. *Basic κ-open* sets are the sets of the form

$$N_\eta = \{\zeta \in \kappa^\kappa \mid \eta \subseteq \zeta\}$$

where $\eta : X \to \kappa$ (or $\zeta : \kappa \to \{0,1\}$ and $\eta : X \to \{0,1\}$ in the case of 2^κ) and $X \subseteq \kappa$ has size less than κ; and the empty set. A set is (κ,λ)-open if it is a λ-union of (i.e. a union of λ) basic κ-*open* sets. A set is (κ,λ)-*closed* if its complement is (κ,λ)-open. The class of (κ,λ)-*Borel* sets is the smallest class that contains all the basic open sets and is closed under complements and λ-unions and λ-intersections. Sets that are (κ,κ)-open (-closed, -Borel) are called simply κ-open (-closed,-Borel) sets.

Note that the total number of basic κ-open sets is $\kappa^{<\kappa}$, so $(\kappa,\kappa^{<\kappa})$-open is equivalent to open in the usual sense of Generalized Baire Spaces, as is κ-open if $\kappa^{<\kappa} = \kappa$. On the other hand, if $\kappa^{<\kappa} > \kappa$, then we get a difference:

Proposition 2.4. *Suppose $2^\omega > \kappa$. Then there is an open κ-Borel set which is not κ-open.*

Proof. Let X be the set of all strictly increasing functions $f : \omega \to \kappa$ and $U = \bigcup_{f\in X} N_f$. It is clear that U is open. Since obviously $U = \bigcap_{n<\omega}(\bigcup_{f\in X} N_{f\restriction n})$, U is κ-Borel. Let us show that U is not a κ-open.

Assume, towards contradiction, that there is $Z \subseteq \kappa^\kappa$ such that $|Z| \leq \kappa$ and $U = \bigcup_{\eta \in Z} N_\eta$. Notice that for all $\eta \in Z$, $\omega \subseteq dom(\eta)$, otherwise there is $\xi \in N_\eta$ such that $\xi \restriction \omega$ is not strictly increasing. It is easy to see that $|X| = \kappa^\omega > \kappa$, so there is $f \in X$ such that $f \not\subseteq \eta$, for all $\eta \in Z$. Thus $N_f \not\subseteq \bigcup_{\eta \in Z} N_\eta = U$, a contradiction. $\boxminus$

Notice that 2^κ is a κ-Borel subset of κ^κ. This observation will be useful later.

We define basic κ-open sets in $\kappa^\kappa \times \kappa^\kappa$ in a similar way. There the basic κ-open sets are of the form $N_\eta \times N_\xi$, where N_η and N_ξ are basic κ-open sets of κ^κ. A set is (κ, λ)-open in $\kappa^\kappa \times \kappa^\kappa$ if it is a λ-union of basic κ-open sets. A set is (κ, λ)-closed if its complement is (κ, λ)-open. The class of (κ, λ)-Borel sets is the smallest class that contains all the basic open sets and is closed under complements, λ-unions, and λ-intersections.

3. κ-Borel and κ-Borel*

We will redefine Borel* sets anew as well. However, there is no change if $\kappa^{<\kappa} = \kappa$. In classical Polish spaces every Borel set has a Borel-code which is a countable tree without infinite branches. Borel*-sets ([8]) arise when the Borel-code is allowed to have infinite branches. We first define the appropriate trees and then define what it means for such a tree to be a Borel*-code.

Definition 3.1. A pair (T, L) is a *good labelled* (κ, λ)*-tree* if

(i) $T = (T, \leq)$ is a tree without κ-branches.
(ii) Every element of T has $\leq \kappa$ many immediate successors.
(iii) $|T| \leq \lambda$.
(iv) Every increasing sequence in T has a supremum.
(v) If $t \in T$ is not a leaf (i.e. maximal element), then $L(t) \in \{\bigcup, \bigcap\}$.
(vi) If $t \in T$ is a leaf, then $L(t)$ is a basic κ-open set.

Sometimes we use (κ, λ)-Borel sets as labels. It is clear how this fits in the general definition.

Definition 3.2. A good labelled (κ, λ)-tree is a (κ, λ)*-Borel***-code* if

(i) There is a function π such that $\mathrm{dom}(\pi)$ is a κ-closed subset of κ^κ and for all $\eta \in \mathrm{dom}(\pi)$, $\pi(\eta)$ is a strategy of II in the usual (see [8], [3]) Borel*-game $GB(\xi, (T, L))$, where $\xi \in \kappa^\kappa$.
(ii) For all $\xi \in \kappa^\kappa$, if II has a winning strategy in $GB(\xi, (T, L))$, denoted $II \uparrow GB(\xi, (T, L))$, then there is some $\eta \in \mathrm{dom}(\pi)$ such that $\pi(\eta)$ is a winning strategy of II in $GB(\xi, (T, L))$.
(iii) The set

$$\{(\xi, \eta) : \eta \in \mathrm{dom}(\pi) \text{ and } \pi(\eta) \text{ is a winning strategy of II in } GB(\xi, (T, L))\}$$

is κ-Borel.

Now that we have defined (κ,λ)-Borel*-codes, we can define (κ,λ)-Borel*-sets as such sets that have a (κ,λ)-Borel*-code:

Definition 3.3. A set $X \subseteq \kappa^\kappa$ is (κ,λ)-*Borel** if there is a (κ,λ)-Borel*-code (T,L) such that for all $\xi \in \kappa^\kappa$:

$$\xi \in X \leftrightarrow II \uparrow GB(\xi,(T,L)).$$

Lemma 3.4. *Suppose $\lambda \leq \kappa$. If (T,L) is a good labelled (κ,λ)-tree, then it is a (κ,λ)-Borel*-code.*

Proof. Let $X = \{t \in T \mid L(t) = \cup\}$ and $F : X \to \kappa$ be an injective function. For all $t \in X$, let G_t be an injective function from the immediate successors of t to κ. Let $\mathrm{dom}(\pi)$ be the set of all $\eta \in \kappa^\kappa$ such that for all $t \in X$, $\eta(F(t)) \in \mathrm{rng}(G_t)$. It is clear that $\mathrm{dom}(\pi)$ is κ-closed. Let $\pi(\eta)$ be the strategy that at $t \in X$, it chooses $G_t^{-1}(\eta(F(t)))$. Clearly (i) and (ii) are satisfied.

Let $Y = \{t \in T \mid t \text{ is a leaf}\}$, since $|T| \leq \kappa$, Y has power at most κ. For all $t \in Y$, let η_t be such that $L(t) = N_{\eta_t}$. For all $t \in Y$, there are $Y_t \subseteq \kappa$, $|Y_t| < \kappa$, and $\xi_t : Y_t \to \kappa$ such that for all $\vartheta \in \mathrm{dom}(\pi)$, $\xi_t \subseteq \vartheta$ if and only if for all $\zeta \in \kappa^\kappa$, there is a play in $GB(\zeta,(T,L))$ in which II has used $\pi(\vartheta)$ and it ends at t.

We conclude that for all $(\zeta,\vartheta) \in (\kappa^\kappa \backslash N_{\eta_t}) \times (\mathrm{dom}(\pi) \cap N_{\xi_t})$, $\pi(\vartheta)$ is not a winning strategy of II in $GB(\zeta,(T,L))$. Thus

$$H = \bigcup_{t \in Y} (\kappa^\kappa \backslash N_{\eta_t}) \times (\mathrm{dom}(\pi) \cap N_{\xi_t})$$

is κ-Borel. Notice that

$$\{(\xi,\eta) : \eta \in \mathrm{dom}(\pi) \text{ and } \pi(\eta) \text{ is a winning strategy of II in } GB(\xi,(T,L))\}$$

is equal to $(\kappa^\kappa \times \kappa^\kappa \backslash\, H) \cap (\kappa^\kappa \times \mathrm{dom}(\pi))$, the intersection of two κ-Borel sets. Therefore

$$\{(\xi,\eta) : \eta \in \mathrm{dom}(\pi) \text{ and } \pi(\eta) \text{ is a winning strategy of II in } GB(\xi,(T,L))\}$$

is κ-Borel and (iii) holds.

$\boxminus$

It follows that if $\kappa^{<\kappa} = \kappa$ and $\lambda \geq \kappa$, then (κ,λ)-Borel* implies (κ,κ)-Borel*, and (κ,κ)-Borel* is the same as Borel* in [2] and [8].

Corollary 3.5. *Suppose $\lambda \leq \kappa$ and $X \subseteq \kappa^\kappa$ is a (κ,λ)-*Borel* *set. Then X is the projection of a κ-closed set.*

Proof. By the definition of (κ,λ)-*Borel** set, there is a (κ,λ)-Borel*-code (T,L) such that for all $\xi \in \kappa^\kappa$:

$$\xi \in X \leftrightarrow II \uparrow GB(\xi,(T,L)).$$

By definition, (T, L) is a good labelled (κ, λ)-tree. Let π be the function constructed in Lemma 3.4 to show that (T, L) is a (κ, λ)-Borel*-code. Let us show that

$$\{(\xi, \eta) : \eta \in \mathrm{dom}(\pi) \text{ and } \pi(\eta) \text{ is a winning strategy of II in } GB(\xi, (T, L))\} \tag{3}$$

is κ-closed. Let Y, η_t, and ξ_t as in Lemma 3.4.

Notice that the complement of basic κ-open sets are κ-unions of basic κ-open sets, i.e. are κ-open. Thus the sets

$$(\kappa^\kappa \backslash N_{\zeta_t}) \times N_{\eta_t}$$

from the proof of Lemma 3.4, are κ-unions of κ-open sets. Thus

$$A = \bigcup_{t \in Y} (\kappa^\kappa \backslash N_{\zeta_t}) \times N_{\eta_t}$$

is κ-open.

Notice that (3) is equal to $(\kappa^\kappa \times \kappa^\kappa \backslash\ A) \cap (\kappa^\kappa \times \mathrm{dom}(\pi))$, the intersection of two κ-closed sets. Thus the set (3) is κ-closed. ⊟

Definition 3.6. We say that a set is κ-Σ_1^1 if it is the projection of a κ-Borel set. A set is κ-Π_1^1 if it is the complement of a κ-Σ_1^1. A set is κ-Δ_1^1 if it is κ-Σ_1^1 and κ-Π_1^1

Notice that there are only 2^κ many κ-Σ_1^1 sets, but there are 2^{2^κ} subsets of 2^κ. Hence most subsetes of 2^κ are not κ-Σ_1^1. Sets that are (κ, κ)-Borel*are called simply κ-Borel*. It is clear from Definition 3.2 (iii), that if $X \subseteq \kappa^\kappa$ is κ-Borel*, then it is κ-Σ_1^1.

Definition 3.7. Let us define the following hierarchy.

- κ-Σ_0 is the set of basic κ-open sets of the form $N_{\{(i,j)\}}$.
- If α is even, then κ-$\Sigma_{\alpha+1}$ is the set of all κ-unions of κ-Σ_α-sets.
- If α is odd, then κ-$\Sigma_{\alpha+1}$ is the set of all κ-intersections of κ-Σ_α-sets.
- If α is limit, then κ-$\Sigma_\alpha = \cup_{\beta<\alpha}\kappa$-$\Sigma_\beta$

Clearly κ-Borel$= \bigcup_{\alpha<\kappa^+} \kappa$-$\Sigma_\alpha$.

We call a function $f \colon \kappa^\kappa \to \kappa^\kappa$ κ-continuous if the inverse image of a κ-open set is a κ-open set.

Fact 3.8. *If $f \colon \kappa^\kappa \to \kappa^\kappa$ is a κ-continuous function, then for all κ-Borel $X \subseteq \kappa^\kappa$, $f^{-1}[X]$ is κ-Borel.*

Proof. Let us proceed by induction over κ-Σ_α, from Definition 3.7. Since f is κ-continuous, if $X \in \kappa$-Σ_0, then $f^{-1}[X]$ is κ-open. Thus X is κ-Borel. Let us suppose that $\alpha < \kappa^+$ is such that for all $\beta < \alpha$, if $X \in \kappa$-Σ_β, then $f^{-1}[X]$ is κ-Borel. It is clear that if α is limit, then for all $X \in \kappa$-$\Sigma_\alpha = \cup_{\beta<\alpha}\kappa$-$\Sigma_\beta$, $f^{-1}[X]$ is κ-Borel.

Let us suppose $\alpha = \beta + 1$ for some $\beta < \kappa^+$ even (the odd case is similar). Let $X \in \kappa$-Σ_α. So, $X = \bigcup_{\gamma<\kappa} A_\gamma$, where $A_\gamma \in \kappa$-Σ_β. Since A_γ

is κ-Borel and $f^{-1}[X]=\bigcup_{\gamma<\kappa} f^{-1}[A_\gamma]$, we conclude that X is κ-Borel. $\boxminus$

For all $t\in T$ we will denote by t^- the immediate predecessor of t (in case it exists), i.e. the unique element of T that satisfies:

- $t^- < t$,
- there is no $t''\in T$ such that $t^- < t'' < t$.

Definition 3.9. Let T be a tree without infinite branches. For all $t\in T$, we define $rk(t)$ as follows:

- If t is a leaf, then $rk(t)=0$;
- if t is not a leaf, then $rk(t)=\cup\{rk(t')+1 \mid t'^-=t\}$.
- If T is not empty and has a root, r, then the rank of T is denoted by $rk(T)$ and is equal to $rk(r)$.

Lemma 3.10. *Suppose κ is such that $\kappa^\omega=\kappa$. A set $X\subseteq\kappa^\kappa$ is κ-Borel if and only if there is a κ-Borel*-code (T,L) such that T is a subtree of $\kappa^{<\omega}$ and for all $\xi\in\kappa^\kappa$:*

$$\xi\in X \leftrightarrow II\uparrow GB(\xi,(T,L)).$$

Proof. $\Rightarrow$) We will show by induction over α that for every $X\in\kappa\text{-}\Sigma_\alpha$, the statement holds.

The result is trivial when $\alpha=0$.

Suppose α is such that for all $\beta\leq\alpha$ and $X\in\kappa\text{-}\Sigma_\beta$, there is a κ-Borel*-code (T,L) such that (T,L) codes X and T is a subtree of $\kappa^{<\omega}$. Let us show the case where α is even. The odd case is similar. Suppose $X\in\kappa\text{-}\Sigma_{\alpha+1}$, so $X=\bigcup_{\gamma<\kappa}A_\gamma$, where $A_\gamma\in\kappa\text{-}\Sigma_\alpha$. By the induction hypothesis, there are κ-Borel*-codes $\{(T_\gamma,L_\gamma)\}_{\gamma<\kappa}$ such that (T_γ,L_γ) codes A_γ and T_γ is a subtree of $\kappa^{<\omega}$, for all γ. Let $\mathcal{T}=\{r\}\cup\bigcup_{\gamma<\kappa}T_\gamma\times\{\gamma\}$ be the tree ordered by:

- $r<_{\mathcal{T}}(x,j)$ for all $(x,j)\in\bigcup_{\gamma<\kappa}T_\gamma\times\{\gamma\}$,
- $(x,\gamma)<_{\mathcal{T}}(y,j)$ if and only if $\gamma=j$ and $x<_{T_\gamma}y$ in T_γ.

Let $T\subseteq\kappa^{<\omega}$ be a tree isomorphic to $\mathcal{T}$ and let $\mathcal{G}:T\to\mathcal{T}$ be a tree isomorphism. If $\mathcal{G}(x)\neq r$, then denote $\mathcal{G}(x)$ by $(\mathcal{G}_1(x),\mathcal{G}_2(x))$. Define L by $L(x)=\cup$ if $G(x)=r$, and $L(x)=L_{\mathcal{G}_2(x)}(\mathcal{G}_1(x))$.

Let us show that (T,L) codes X. Let $\eta\in X$, so there is $\gamma<\kappa$, such that $\eta\in X_\gamma$. II starts by choosing $\mathcal{G}^{-1}(x,\gamma)$, where x is the root of T_γ. II continues playing with the winning strategy from the game $GB(\eta,(T_\gamma,L_\gamma))$, choosing the element given by $\mathcal{G}^{-1}$. We conclude that $II\uparrow GB(\eta,(T,L))$.

Let $\eta\notin X$, so for all $\gamma<\kappa$, $\eta\notin X_\gamma$, so II has no winning strategy for the game $GB(\eta,(T,L))$. Thus II cannot have a winning strategy for the game $GB(\eta,(T,L))$.

Let us code the winning strategies by a function π, such that the set

$$W=\{(\xi,\eta):\pi(\eta)\text{ is a winning strategy of II in }GB(\xi,(T,L))\}$$

is κ-Borel.

Since for all $\gamma < \kappa$, (T_γ, L_γ) is a κ-Borel*-code, there is a function π_γ such that

$$W_\gamma = \{(\xi,\eta) : \pi_\gamma(\eta) \text{ is a winning strategy of II in } GB(\xi,(T_\gamma,L_\gamma))\}$$

is κ-Borel. Let us define $p : \kappa^\kappa \to \kappa^\kappa$ by

$$p(\eta)(\alpha) = \begin{cases} \eta(\alpha) & \text{if } \alpha \geq \omega, \\ \eta(\alpha+1) & \text{if } \alpha < \omega. \end{cases}$$

Let $\pi(\eta)$ be the strategy in which II starts by choosing $\eta(0)$ and then plays following the strategy $\pi_{\eta(0)}(p(\eta))$. Let g be the map $(\xi,\eta) \mapsto (\xi, p(\eta))$. Clearly p is κ-continuous, thus g is κ-continuous. Therefore $g^{-1}[W_\gamma]$ is κ-Borel.

Notice that $\pi(\eta)$ is a winning strategy of II in $GB(\xi,(T,L))$ if and only if $\pi_{\eta(0)}(p(\eta))$ is a winning strategy of II in $GB(\xi,(T_{\eta(0)},L_{\eta(0)}))$. Since $W = \bigcup_{\gamma<\kappa} g^{-1}[W_\gamma]$ we conclude that W is κ-Borel.

The case α limit is similar to the previous one, since $\kappa\text{-}\Sigma_\alpha = \cup_{\beta<\alpha}\kappa\text{-}\Sigma_\beta$ when α is limit.

$\Leftarrow$) We will use induction over the rank of T, to show that if (T,L) is a good labelled (κ,κ)-tree such that T is a subtree of $\kappa^{<\omega}$, then the set X such that for all $\xi \in \kappa^\kappa$:

$$\xi \in X \leftrightarrow II \uparrow GB(\xi,(T,L)),$$

is a κ-Borel set. Notice that this implies that if (T,L) is a κ-Borel*-code where T is a subtree of $\kappa^{<\omega}$, then codes a κ-Borel set.

If $rk(T) = 0$, then T has only one node r, thus $X = L(r)$ and X is a basic κ-open set. Let $\alpha < \kappa^+$ be such that for all good labelled (κ,κ)-tree (T',L') with T' a subtree of $\kappa^{<\omega}$ and $rk(T') < \alpha$, (T',L') codes a κ-Borel set.

Let (T,L) be a good labelled (κ,κ)-tree such that T is a subtree of $\kappa^{<\omega}$, with $rk(T) = \alpha$, and X is such that for all $\xi \in \kappa^\kappa$:

$$\xi \in X \leftrightarrow II \uparrow GB(\xi,(T,L)).$$

Let $B = \{t \in T \mid t^- = r\}$, where r is the root of T. Notice that $|B| \leq \kappa$ by Definition 3.1 (ii). For all $t \in B$, define the good labelled (κ,κ)-tree (T_t, L_t) as follows:

- $T_t = \{x \in T \mid t \leq x\}$,
- $L_t = L \restriction T_t$.

Since $rk(T) = \alpha$, for all $t \in B$, $rk(T_t) < \alpha$. By the induction hypothesis, for all $t \in B$ the set X_t of all $\xi \in \kappa^\kappa$ such that $II \uparrow GB(\xi,(T_t,L_t))$, is a κ-Borel set.

It is easy to see that, if $L(r) = \cup$, then $X = \cup_{t\in B} X_t$. Also, if $L(r) = \cap$, then $X = \cap_{t\in B} X_t$.

Since the class of κ-Borel sets is closed under unions and intersections of length κ, the proof follows. $\boxminus$

Lemma 3.11. *Every κ-Borel set is the projection of a κ-closed set (and hence κ-Σ^1_1).*

Proof. It follows from Corollary 3.5 and Lemma 3.10. $\boxminus$

Corollary 3.12. *κ-Σ^1_1 sets are projections of κ-closed sets.*

Proof. Let A be a κ-Σ^1_1 set. Thus, A is the projection of a κ-Borel set B. By the previous Lemma, B is the projection of a κ-closed set. Thus A is the projection of a κ-closed set. $\boxminus$

Definition 3.13.

- The class of *weak κ-Borel* sets is the smallest class that contains all the open sets and is closed under complements, κ-unions, and κ-intersections.
- We define *weak good labelled κ-trees* as in Definition 3.1 by omitting (iii).
- We define *weak κ-Borel***-codes* as in Definition 3.2 by using weak good labelled κ-trees.
- We define *weak κ-Borel** sets as in Definition 3.3 using weak κ-Borel*-codes.

Lemma 3.14. *Suppose $2^\omega = 2^\kappa$, where $\kappa > \omega$. Then every $A \subseteq \kappa^\kappa$ is a weak κ-Borel** *set.*

Proof. Let $\Pi : \kappa^\omega \to \kappa^\kappa$ be a bijection and $A \subseteq \kappa^\kappa$. Let $T = \kappa^{\leq\omega+1}$, and define L as follows:

- $L(\eta) = \cup$ if $dom(\eta) < \omega$;
- $L(\eta) = \cap$ if $dom(\eta) = \omega$;
- If $dom(\eta) = \omega + 1$, then

$$L(\eta) = \begin{cases} N_{\Pi(\eta\restriction\omega)\restriction\eta(\omega)} & \text{if } \Pi(\eta\restriction\omega) \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly (T, L) is a weak good labelled κ-tree. Let us show that

$$\xi \in A \leftrightarrow II \uparrow GB(\xi, (T, L)).$$

Let ξ be such that $II \uparrow GB(\xi, (T, L))$, then there is $\eta \in T$ with $dom(\eta) = \omega + 1$, such that $\xi \in L(\eta)$. Thus $L(\eta) \neq \emptyset$ and $\Pi(\eta\restriction\omega) \in A$. Clearly $\xi \in \bigcap_{\alpha<\kappa} N_{\Pi(\eta\restriction\omega)\restriction\alpha}$ if and only if $\xi = \Pi(\eta\restriction\omega)$. Since $\Pi(\eta\restriction\omega) \in A$, $\xi \in A$.

Let $\xi \in A$. There is $\eta \in \kappa^\omega$ such that $\Pi(\eta) = \xi$. II could play such that after ω moves the game is at η. Since $\Pi(\eta) = \xi$, it doesn't matter where I moves, II will win.

Let $\pi(\zeta)$ be the strategy in which II copies $\zeta\restriction\omega$. Let us show that

$$W = \{(\xi, \zeta) : \pi(\zeta) \text{ is a winning strategy of II in } GB(\xi, (T, L))\}$$

is a weak κ-Borel set. Notice that $W = \{(\xi, \zeta) \mid \xi = \Pi(\zeta\restriction\omega)\}$. Let $\xi, \zeta \in \kappa^\kappa$ be such that $(\xi, \zeta) \notin W$. Thus $\xi \neq \Pi(\zeta\restriction\omega)$. Let $\alpha < \kappa$

be such that $\xi(\alpha) \neq \Pi(\zeta \restriction \omega)(\alpha)$. Thus for all $(\xi', \zeta') \in N_{\xi\restriction\{\alpha\}} \times N_{\zeta\restriction\omega}$, $(\xi', \zeta') \notin W$. We conclude that W is closed.

$\boxminus$

Notice that (T, L) in the previous lemma is not a κ-Borel*-code.

Lemma 3.15. *If $X \subseteq \kappa^\kappa$ is κ-Δ_1^1, then it is $(\kappa, \kappa^{<\kappa})$-Borel*.*

Proof. Let $X \subseteq \kappa^\kappa$ be a κ-Δ_1^1. Let Y and Z be κ-closed such that $X = pr_1(Y)$ and $\kappa\backslash X = pr_1(Z)$.

Let T consist of triples (x_0, x_1, x_2) that satisfies one of the following:

- $(\eta_0, \eta_1, \eta_2) \in (\kappa^{<\kappa})^3$ such that one of the following holds:
  - $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_1) = \mathrm{dom}(\eta_2)$;
  - $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_1) = \mathrm{dom}(\eta_2) + 1$.

  The triples are ordered by coordinate wise inclusion. A triplet $(\eta_0', \eta_1', \eta_2')$ is not a leaf if and only if $N_{(\eta_0', \eta_1')} \cap Z \neq \emptyset$ and $N_{(\eta_0', \eta_2')} \cap Y \neq \emptyset$.
- $(\eta_0, \eta_2, \alpha) \in (\kappa^{<\kappa})^2 \times \kappa$ such that $\alpha \neq 0$ and one of the following holds:
  - $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_2) \leq \alpha$,
  - $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_2) + 1 \leq \alpha$.

  The triples are ordered by $(\eta_0, \eta_2, \alpha) \leq (\eta_0', \eta_2', \beta)$ if $\eta_0 \subseteq \eta_0'$, $\eta_2 \subseteq \eta_2'$, and $\alpha = \beta$. If $(\eta_0', \eta_2', \alpha)$ is not a leaf, then $N_{(\eta_0', \eta_2')} \cap Y \neq \emptyset$.

Finally we say that $(\emptyset, \emptyset, \emptyset) \leq (x_0, x_1, x_2)$ for all the triples, independent of which kind it is. Let us define the labelled function, L, for T:

- Case $t = (\eta_0, \eta_1, \eta_2) \in (\kappa^{<\kappa})^3$:
  - If t is not a leaf and $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_2)$, then $L(t) = \cap$. If t is not a leaf and $\mathrm{dom}(\eta_0) \neq \mathrm{dom}(\eta_2)$, then $L(t) = \cup$.
  - If t is a leaf and $N_{(\eta_0, \eta_1)} \cap Z = \emptyset$, then $L(t) = \kappa^\kappa$. If t is a leaf and $N_{(\eta_0, \eta_1)} \cap Z \neq \emptyset$, then $L(t) = \kappa^\kappa \backslash N_{\eta_0}$.
- Case $t = (\eta_0, \eta_2, \alpha) \in (\kappa^{<\kappa})^2 \times \kappa$:
  - If t is not a leaf and $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_2)$, then $L(t) = \cap$. If t is not a leaf and $\mathrm{dom}(\eta_0) \neq \mathrm{dom}(\eta_2)$, then $L(t) = \cup$.
  - If t is a leaf and $N_{(\eta_0, \eta_2)} \cap Y = \emptyset$, then $L(t) = \kappa^\kappa \backslash N_{\eta_0}$. If t is a leaf and $N_{(\eta_0, \eta_2)} \cap Y \neq \emptyset$, then $L(t) = \kappa^\kappa$.

Let us show that T has no κ-branch. Assume, for sake of contradiction, that T has a κ-branch. By the construction of T, there is $(\eta_0, \eta_1, \eta_2) \in \kappa^\kappa$ such that for all $\beta < \kappa$, $(\eta_0 \restriction \beta, \eta_1 \restriction \beta, \eta_2 \restriction \beta) \in T$. Thus, for all $\beta < \kappa$, $N_{(\eta_0\restriction\beta, \eta_1\restriction\beta)} \cap Z \neq \emptyset$ and $N_{(\eta_0\restriction\beta, \eta_2\restriction\beta)} \cap Y \neq \emptyset$. Since Z and Y are κ-closed, $(\eta_0, \eta_1) \in Z$ and $(\eta_0, \eta_2) \in Y$. So $\eta_0 \in Y \cap Z$, a contradiction.

Notice that T is closed under coordinate wise increasing unions, so very increasing sequence in T has a supremum. Since $|T| \leq \kappa^{<\kappa}$, (T, L) is a good labelled $(\kappa, \kappa^{<\kappa})$-tree.

Let us show that for all $\xi \in \kappa^\kappa$, $\xi \in X$ if and only if $II \uparrow GB(\xi, (T, L))$. For all $\eta \in \kappa^\kappa$, let $\pi(\eta)$ be the strategy in which II chooses initial segments of η, i.e. if the game is at $(\eta'_0, \eta'_1, \eta'_2)$ such that $L((\eta'_0, \eta'_1, \eta'_2)) = \cup$, II chooses the unique triple (η_0, η_1, η_2) that satisfies $\mathrm{dom}(\eta_0) = \mathrm{dom}(\eta_1) = \mathrm{dom}(\eta_2)$, $\eta'_0 = \eta_0$, $\eta'_1 = \eta_1$, and $\eta'_2 \subseteq \eta_2 \subseteq \eta$ (the same idea applies if the game is at $(\eta'_0, \eta'_2, \alpha)$).

Let $\xi \in \kappa^\kappa$ be such that $\xi \in X$. So, there is $\eta \in \kappa^\kappa$ such that $(\xi, \eta) \in Y$. Suppose that II plays following $\pi(\eta)$ and the game ends at the leaf (x_0, x_1, x_2).

- Case $(x_0, x_1, x_2) = (\eta_0, \eta_1, \eta_2) \in (\kappa^{<\kappa})^3$: If $\eta_0 \not\subseteq \xi$, then $\xi \in \kappa^\kappa \backslash N_{\eta_0}$. Thus $\xi \in L((\eta_0, \eta_1, \eta_2))$ and it doesn't matter whether $N_{(\eta_0, \eta_1)} \cap Z = \emptyset$ or not. On the other hand, by the definition of $\pi(\eta)$, $\eta_2 \subseteq \eta$. Thus, if $\eta_0 \subseteq \xi$, then $N_{(\eta'_0, \eta'_2)} \cap Y \neq \emptyset$ and $N_{(\eta'_0, \eta'_1)} \cap Z = \emptyset$. So $\xi \in L((\eta_0, \eta_1, \eta_2))$.
- Case $(x_0, x_1, x_2) = (\eta_0, \eta_2, \alpha) \in (\kappa^{<\kappa})^2 \times \kappa$: If $N_{(\eta_0, \eta_2)} \cap Y \neq \emptyset$, then $\xi \in \kappa^\kappa = L((\eta_0, \eta_2, \alpha))$. On the other hand, by the definition of $\pi(\eta)$, $\eta_2 \subseteq \eta$. Thus, if $N_{(\eta_0, \eta_2)} \cap Y = \emptyset$, then $\eta_0 \not\subseteq \xi$. So $\xi \in \kappa^\kappa \backslash N_{\eta_0} = L((\eta_0, \eta_1, \eta_2))$.

We conclude that $\pi(\eta)$ is a winning strategy of II in $GB(\xi, (T, L))$.

For all $\xi, \eta \in (\kappa^\kappa)^2$, let $\sigma(\xi, \eta)$ be the strategy in which I chooses initial segments of (ξ, η). If $\xi \notin X$, then there is $\eta \in \kappa^\kappa$ such that $(\xi, \eta) \in Z$. Clearly $\sigma(\xi, \eta)$ is a winning strategy of I in $GB(\xi, (T, L))$.

We conclude that

$$\xi \in X \leftrightarrow II \uparrow GB(\xi, (T, L)).$$

We will finish the prove by showing that

$$W = \{(\xi, \eta) : \pi(\eta) \text{ is a winning strategy of II in } GB(\xi, (T, L))\}$$

is κ-Borel. From our previous argument, if $(\xi, \eta) \in Y$, then $(\xi, \eta) \in W$. Let $(\xi, \eta) \notin Y$, then there is $\alpha < \kappa$ such that $(N_{\xi\restriction\alpha} \times N_{\eta\restriction\alpha}) \cap Y = \emptyset$. If I starts by choosing first $(\emptyset, \emptyset, \alpha)$, then $(\xi \restriction 1, \emptyset, \alpha)$, and then, in each turns chooses an initial segment of ξ, then II will lose if II follows $\pi(\eta)$. Thus $\pi(\eta)$ is not a winning strategy. So $(\xi, \eta) \notin Y$ implies $\pi(\eta) \notin W$. Therefore $Y = W$. Since Y is κ-closed, W is κ-Borel.

$\boxminus$

Let X be a subclass of κ-Σ^1_1. We say that a set $U \subseteq \kappa^\kappa \times \kappa^\kappa$ is *universal for* X if for all $A \in X$, there is $\xi \in \kappa^\kappa$ such that for all $\eta \in \kappa^\kappa$, $\eta \in A$ if and only if $(\eta, \xi) \in U$.

Lemma 3.16. *For all $\alpha < \kappa^+$ there is $R_\alpha \subseteq \kappa^\kappa \times \kappa^\kappa$ which is a κ-Borel set and universal for κ-Σ_α.*

Proof. We will proceed by induction over α.

$\alpha = 0$: Let Z be the set of tuples $(\{(i, j)\}, \eta)$ where $i, j \in \kappa$, $\mathrm{dom}(\eta) = i + 1$, for all $k < i$, $\eta(k) = 0$, and $\eta(i) = j + 1$. Let $R_0 = \bigcup_{z \in Z} N_z$, it is clear that R_0 is κ-open and universal for κ-Σ_0

$\alpha+1$**:** Let us show the case where α is even, the odd case is similar. For all $t \in \kappa^{<\omega}$ different from $\emptyset$, let $p(t)$ be such that $\mathrm{dom}(p(t)) = \mathrm{dom}(t) - 1$ and $p(t)(n) = t(n+1)$.

Let α be even such that R_α exists and (T_α, L_α) is the κ-Borel*-code for R_α, where $T \subset \kappa^<\omega$. Let $\Pi : \kappa \times \kappa \to \kappa$ be a bijection such that if $\beta < \beta'$, then $\Pi(\gamma, \beta) < \Pi(\gamma, \beta')$ for all γ.

For all $\gamma < \kappa$, $X \subseteq \kappa$, $|X| \leq \kappa$, and $\eta : X \to \kappa$, let $p_\gamma(\eta) = \xi$ be such that $\mathrm{dom}(\xi) = \{\beta < \kappa \mid \Pi(\gamma, \beta) \in \mathrm{dom}(\eta)\}$, and for all $\beta \in \mathrm{dom}(\xi)$, $\xi(\beta) = \eta(\Pi(\gamma, \beta))$.

For all $\gamma < \kappa$, $X \subseteq \kappa$, $|X| \leq \kappa$, and $\eta : X \to \kappa$, let $p^*_\gamma(\eta) = \xi^*$ be such that $\mathrm{dom}(\xi^*) = \{\Pi(\gamma, \beta) \mid \beta \in \mathrm{dom}(\eta)\}$, and for all $\Pi(\gamma, \beta) \in \mathrm{dom}(\xi^*)$, $\xi^*(\Pi(\gamma, \beta)) = \eta(\beta)$.

Let $T_{\alpha+1}$ is the set of all $t \in \kappa^{<\omega}$ such that $p(t) \in T_\alpha$. $L_{\alpha+1}$ be such that $L_{\alpha+1}(\emptyset) = \cup$; if $t \neq \emptyset$ and it is not a leaf, then $L_{\alpha+1}(t) = L_\alpha(p(t))$; and if t is a leaf and $L_\alpha(p(t)) = N_{(\eta,\xi)}$, then $L_{\alpha+1}(t) = N_{(\eta, p^*_\gamma(\xi))}$, where $\gamma = t(0)$. By Lemma 3.4 it is easy to see that $(T_{\alpha+1}, L_{\alpha+1})$ is a κ-Borel*-code.

Let $R_{\alpha+1}$ be the set whose κ-Borel*-code is $(T_{\alpha+1}, L_{\alpha+1})$. Then $R_{\alpha+1}$ is κ-Borel by Lemma 3.10.

Claim 3.16.1. *$R_{\alpha+1}$ is universal for κ-$\Sigma_{\alpha+1}$.*

Proof. Let $A \in \kappa$-$\Sigma_{\alpha+1}$ and $A = \bigcup_{i<\kappa} A_i$ where for all $i < \kappa$, $A_i \in \kappa$-Σ_α. By the induction hypothesis, for all $i < \kappa$, there is ξ_i such that $\eta \in A_i$ if and only if $(\eta, \xi_i) \in R_\alpha$. It is easy to see that there is ξ such that for all $i < \kappa$, $p_i(\xi) = \xi_i$. From the definition of p_i and p^*_i, ξ exists. Let us show that for all $\eta \in \kappa^\kappa$, $\eta \in A$ if and only if $(\eta, \xi) \in R_{\alpha+1}$. From the way $R_{\alpha+1}$ was constructed, $(\eta, \xi) \in R_{\alpha+1}$ if and only if $II \uparrow GB((\eta, \xi), (T_{\alpha+1}, L_{\alpha+1}))$. Thus it is enough to show that for all $\eta \in \kappa^\kappa$, $\eta \in A$ if and only if $II \uparrow GB((\eta, \xi), (T_{\alpha+1}, L_{\alpha+1}))$.

Let η be such that $\eta \in A$. Therefore, there is $i < \kappa$ such that $\eta \in A_i$ and $(\eta, \xi_i) \in R_\alpha$. Thus $II \uparrow GB((\eta, \xi_i), (T_\alpha, L_\alpha))$, let σ be a winning strategy of II in $GB((\eta, \xi_i), (T_\alpha, L_\alpha))$. Let II plays by the following strategy in $GB((\eta, \xi), (T_{\alpha+1}, L_{\alpha+1}))$: II chooses $(0, i)$ in the first move (since $L_{\alpha+1}(\emptyset) = \cup$) and afterwards, II plays by the strategy σ. By following this strategy, II makes sure that the game ends in a leaf t such that $i = t(0)$, $L_{\alpha+1}(t) = N_{(\zeta_1, p^*_i(\zeta_2))}$, where $L_\alpha(p(t)) = N_{(\zeta_1,\zeta_2)}$, and $(\eta, \xi_i) \in N_{(\zeta_1,\zeta_2)}$. By the way ξ was defined, $(\eta, \xi) \in N_{(\zeta_1, p^*_i(\zeta_2))}$.

Let η be such that $II \uparrow GB((\eta, \xi), (T_{\alpha+1}, L_{\alpha+1}))$. Let σ be a winning strategy of II in $GB((\eta, \xi), (T_{\alpha+1}, L_{\alpha+1}))$. Since $L_{\alpha+1}(\emptyset) = \cup$, there is a unique $\gamma < \kappa$ such that if II plays following σ, then the game ends in a leaf t such that $\gamma = t(0)$, $L_{\alpha+1}(t) = N_{(\zeta_1, p^*_\gamma(\zeta_2))}$, and $(\eta, \xi) \in N_{(\zeta_1, p^*_\gamma(\zeta_2))}$. By the way ξ and $L_{\alpha+1}$ were defined, $L_\alpha(p(t)) = N_{(\zeta_1,\zeta_2)}$, and $(\eta, \xi_\gamma) \in N_{(\zeta_1,\zeta_2)}$. Thus σ induces a winning strategy of II in $GB((\eta, \xi_\gamma), (T_\alpha, L_\alpha))$. We conclude that $(\eta, \xi_\gamma) \in R_\alpha$ and $\eta \in A_\gamma$. Thus $\eta \in A$. $\square$

α **limit:** Let α be such that for all $\beta < \alpha$, such R_β exists. For all $\beta < \alpha$, let (T_β, L_β) be the κ-Borel*-code for R_β, such that T_β is a subtree of $\kappa^{<\omega}$.

Let T_α be the tree of those $t \in \kappa^{<\omega}$ such that $t = \emptyset$ or $t(0) < \alpha$, and $p(t) \in T_{t(0)}$. Let L_α be such that $L_\alpha(\emptyset) = \cup$ and for $t \neq \emptyset$, $L_\alpha(t) = L_{t(0)}(p(t))$. It is easy to see that (T_α, L_α) is a κ-Borel*-code. Let R_α be the set whose κ-Borel*-code is (T_α, L_α). In a similar way as in the previous claim, it is possible to prove that R_α is universal for κ-Σ_α.

$\boxminus$

Corollary 3.17. *The sets κ-Σ_α form a proper hierarchy.*

Proof. Assume, for sake of contradiction, that there is $\alpha < \kappa^+$ such that every κ-Borel set is a κ-Σ_α set. Let $A := \{\eta \in \kappa^\kappa \mid (\eta, \eta) \notin R_\alpha\}$. Clearly $A = (\kappa^\kappa \backslash R_\alpha) \cap \{(\eta, \eta) \mid \eta \in \kappa^\kappa\}$. Since R_α and $\{(\eta, \eta) \mid \eta \in \kappa^\kappa\}$ are κ-Borel, A is κ-Borel. Thus A is κ-Σ_α. On the other hand, since R_α is universal for κ-Σ_α, there is $\xi \in \kappa^\kappa$ such that $\eta \in A$ if and only if $(\eta, \xi) \in R_\alpha$. Thus $(\xi, \xi) \in R_\alpha$ holds if and only if $\xi \in A$. So $(\xi, \xi) \in R_\alpha$ if and only if $(\xi, \xi) \notin R_\alpha$, a contradiction. $\boxminus$

We define the κ-Borel rank of a κ-Borel set, A, as the least ordinal α, such that $A \in \kappa$-Σ_α. We denote the κ-Borel rank of A by $rk_B(A)$.

Theorem 3.18. *There is a (κ, κ^+)-Borel set $R \subseteq \kappa^\kappa \times \kappa^\kappa$ which is universal for κ-Borel sets ((κ, κ)-Borel sets).*

Proof. For all $\alpha < \kappa^+$, let $R_\alpha \subseteq \kappa^\kappa \times \kappa^\kappa$ be a κ-Borel that is universal for κ-Σ_α. Let $\Pi : \kappa^+ \to \kappa^\kappa$ be injective and let us denote $\Pi(\alpha)$ by ξ_α. Let us define R^* by

$$R^* := \bigcup_{\alpha < \kappa^+} R_\alpha \times \{\xi_\alpha\} \subseteq \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa.$$

Since

$$R_\alpha \times \{\xi_\alpha\} = (R_\alpha \times \kappa^\kappa) \cap (\kappa^\kappa \times \kappa^\kappa \times \{\xi_\alpha\}),$$

$R_\alpha \times \{\xi_\alpha\}$ is (κ, κ)-Borel, thus it is a (κ, κ^+)-Borel set. Therefore, R^* is a (κ, κ^+)-Borel set.

It is easy to see that there is $X \subset \kappa$ and $g : X \to \kappa$ a bijection, such that for all $\alpha < \kappa^+$ there is $\bar{R}_\alpha \subseteq \kappa^\kappa \times \kappa^\kappa$, such that for all $A \in \kappa$-Σ_α, $\eta \in A$ if and only if there is ξ such that $(\eta, \xi) \in \bar{R}_\alpha$ and $\xi \restriction X = \xi_\alpha \circ g$. Thus R^* can be coded by a (κ, κ^+)-Borel set universal for (κ, κ)-Borel sets. $\boxminus$

Corollary 3.19. *There is a (κ, κ^+)-Borel set that is not κ-Borel.*

Proof. It can be proved in a similar way as in Corollary 3.17. $\boxminus$

4. Orbits of uncountable models: the non-structure case

Having established, in Sections 2 and 3 above, the basic topological properties of the Generalized Baire Space κ^κ, in the absence of $\kappa^{<\kappa}=\kappa$, we turn in this section to applications in model theory.

Every structure of infinite cardinality κ in a countable vocabulary can be canonically associated with an element of κ^κ, or alternatively an element of 2^κ, leading to close connections between model theoretic properties of structures and topological properties of subsets of κ^κ. To make this translation completely explicit, we make the following definition:

Definition 4.1. Let $L=\{Q_m \mid m\in\omega\}$ be a countable relational language. Fix a bijection π between $\kappa^{<\omega}$ and κ. For every $\eta\in\kappa^\kappa$ define the structure M_η with domain κ as follows: For every tuple $(a_1,a_2,\dots,a_n)$ in κ^n we define

$$(a_1,a_2,\dots,a_n)\in Q_m^{M_\eta} \quad\Leftrightarrow\quad \begin{array}{l} Q_m \text{ has arity } n \text{ and}\\ \eta(\pi(m,a_1,a_2,\dots,a_n))>0.\end{array}$$

Let L and π be as in Definition 4.1. We say that a set $A\subseteq\kappa^\kappa$ is *invariant under* (L,π), if $\eta\in A$ and $M_\eta\cong M_{\eta'}$ imply $\eta'\in A$. Canonical examples of invariant sets are orbits of models M of size κ:

$$\mathrm{Orb}(M)=\{\eta\in\kappa^\kappa : M\cong M_\eta\}.$$

In the realm of countable structures, every orbit is Borel and a subset of ω^ω is invariant Borel if and only it is the set of codes of countable models of a sentence of $L_{\omega_1\omega}$ ([10], [6]). The purpose of this and the next section is to prove a similar result for structures of cardinality $\kappa>\omega$. We show that the answer to this question depends very much on the stability theoretic properties of the first order theory of the structure. Previous work (e.g. [2]) has focused on the case $\kappa^{<\kappa}=\kappa$. Here we do not make this assumption.

The following lemma can be proved exactly as [2, Theorem 24]. That theorem, as well as its proof, are due to R. Vaught, see [2] and [13]. Notice that it seems as if the below lemma solves the "Open problem" at the end of subsection III.1 of [2]. However, in the "Open problem", the notion of a Borel set is different from (κ,κ)-Borel in Lemma 4.2.

Lemma 4.2. *Suppose $\kappa>\omega$ is regular and L and π are as in Definition 4.1. If $A\subseteq\kappa^\kappa$ is κ-Borel and invariant under (L,π), then there is an $L_{\kappa^+\kappa}$-sentence φ such that $A=\{\eta\in\kappa^\kappa \mid M_\eta\models\varphi\}$.*

Definition 4.3. Assume $\mathcal{T}$ a first order theory in a relational countable language, we define the isomorphism relation, $\cong_T\subseteq\kappa^\kappa\times\kappa^\kappa$, as the relation

$$\{(\eta,\xi)|(\mathcal{A}_\eta\models\mathcal{T},\mathcal{A}_\xi\models\mathcal{T},\mathcal{A}_\eta\cong\mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta\not\models\mathcal{T},\mathcal{A}_\xi\not\models\mathcal{T})\}$$

Theorem 4.4. *Suppose $\mathcal{T}$ is a countable complete first order theory. Suppose κ is regular and for all $\alpha < \kappa^+$ there are $\eta, \eta' \in \kappa^\kappa$ such that $M_\eta \models \mathcal{T}$, $M_{\eta'} \models \mathcal{T}$, $M_\eta \equiv^\alpha_{\kappa^+\kappa} M_{\eta'}$, and $M_\eta \not\cong M_{\eta'}$. Then $\cong_T$ is not κ-Borel.*

Proof. Before we start with the proof, let us make some preparations.

Let $\mathcal{T}$ be a countable complete first order theory in a relational countable language, L, and let P be an unary relation symbol not in L. We will work with two different kind of structures, we will work with L-structures and with $L\cup\{P\}$-structures. Let $L = \{Q_m \mid m \in \omega\}$ and let us enumerate $L \cup \{P\}$ by $= \{P_m \mid m \in \omega\}$, where $P = P_0$ and $Q_m = P_{m+1}$. We will use the coding from Definition 4.1 to code L-structures and $L \cup \{P\}$-structures. To avoid misunderstandings, we will denote L-structures different from $L\cup\{P\}$-structures. Let $\eta \in \kappa^\kappa$, we will denote by M_η the L-structure coded by η, and by M^η the $L \cup \{P\}$-structure coded by η. Recall π, the bijection from Definition 4.1.

Let $B = \pi[\{0\} \times \kappa]$ and fix $S_1, S_2 \subseteq B$ disjoint sets of size κ such that $B = S_1 \cup S_2$. For all $\alpha \in B$, we will denote by $\rho(\alpha)$ the projection on the second coordinate of $\pi^{-1}(\alpha)$ (i.e. $\rho(\alpha) = pr_2(\pi^{-1}(\alpha))$).

It is easy to see that $\{\eta \in \kappa^\kappa \mid P^{M^\eta} = \rho[S_1]\}$ is κ-Borel, since it is the κ-intersection of the complement of basic κ-open sets. At the same time, the set of all $\eta \in \kappa^\kappa$ such that for all $0 < m < \omega$ and n-tuple $(a_1, \ldots, a_n) \in \kappa^n \backslash (\rho[S_1]^n \cup \rho[S_2]^n)$, $\eta(\pi(m, a_1 \ldots, a_n)) = 0$, is κ-Borel. Thus the intersection of these two sets is κ-Borel, let us detone it by S.

Let us define $h : S \to 2^\kappa \times 2^\kappa$ as follows. Let $r_1 : \kappa \to \rho[S_1]$ and $r_2 : \kappa \to \rho[S_2]$ be the order preserving bijection.

For all pair $(\eta_1, \eta_2) \in 2^\kappa \times 2^\kappa$ define $h^{-1}[(\eta_1, \eta_2)]$ as the subset of κ^κ of ξ's such that the following hold:

- If $\pi(0, a) \in S_1$, then $\xi(\pi(0, a)) > 0$.
- If $\pi(0, a) \in S_2$, then $\xi(\pi(0, a)) = 0$.
- For all $m < \omega$ and n-tuple $(a_1, \ldots, a_n) \in \kappa^n \backslash (\rho[S_1]^n \cup \rho[S_2]^n)$, $\xi(\pi(m, a_1 \ldots, a_n)) = 0$.
- For any n-tuple $(a_1, \ldots, a_n) \in \kappa^n$ and $m < \omega$,
$$\xi(\pi(m + 1, r_1(a_1) \ldots, r_1(a_n))) > 0$$
if and only if $\eta_1(\pi(m, a_1, \ldots, a_n)) = 1$.
- For any n-tuple $(a_1, \ldots, a_n) \in \kappa^n$ and $m < \omega$,
$$\xi(\pi(m + 1, r_2(a_1) \ldots, r_2(a_n))) > 0$$
if and only if $\eta_2(\pi(m, a_1, \ldots, a_n)) = 1$.

Clearly, for all $\xi \in S$, there is a unique pair $(\eta_1, \eta_2) \in 2^\kappa \times 2^\kappa$ such that $\xi \in h^{-1}[(\eta_1, \eta_2)]$. Since for all $\eta_1, \eta_2 \in 2^\kappa$, $h^{-1}[(\eta_1, \eta_2)] \subseteq S$, h is well defined. Notice that if $h(\xi) = (\eta_1, \eta_2)$, then r_1 is an ismorphism from M_{η_1} to $(M^\xi \cap P^{M^\xi}) \restriction L$ and r_2 is an ismorphism from M_{η_2} to $(M^\xi \backslash P^{M^\xi}) \restriction L$.

Claim 4.4.1. *h is κ-continuous in S, i.e. the inverse image of a κ-open, U, is a κ-open set intersected with S.*

Proof. Let $U = N_p \times N_q$ be a basic κ-open set of $2^\kappa \times 2^\kappa$ and $\xi \in h^{-1}[U]$. Let

$$A_p = \{\pi(m+1, r_1(a_1)\dots, r_1(a_n)) \mid (m, a_1, \dots, a_n) \in \pi^{-1}[\mathrm{dom}(p)]\}$$

and

$$A_q = \{\pi(m+1, r_2(a_1)\dots, r_2(a_n)) \mid (m, a_1, \dots, a_n) \in \pi^{-1}[\mathrm{dom}(q)]\}.$$

Since U is a basic open set, $|\mathrm{dom}(p)|, |\mathrm{dom}(q)| < \kappa$. Thus $|A_p \cup A_q| < \kappa$. Therefore, $N_{\xi \restriction A_p \cup A_q} \cap S \subseteq h^{-1}[U]$, so h is κ-continuous in S. $\boxminus$

Since S is κ-Borel, a variations of Fact 3.8 applies to h: If $Y \subseteq 2^\kappa \times 2^\kappa$ is κ-Borel, then $h^{-1}[Y]$ is κ-Borel.

Notice that h is surjective. We will denote $h(\xi)$ by $(h_1(\xi), h_2(\xi))$.

Let us assume, for sake of contradiction, that $\mathcal{T}$ satisfies the assumptions and $\cong_T$ is κ-Borel. Let

$$X = \{\xi \in S \mid M_{h_1(\xi)} \cong M_{h_2(\xi)}\} = h^{-1}[\cong_T \cap (2^\kappa \times 2^\kappa)],$$

since $2^\kappa \times 2^\kappa$ and $\cong_T$ are κ-Borel, X is κ-Borel. Notice that if η and ξ are such that M^η and M^ξ are isomorphic, then $(M^\eta \cap P^{M^\eta}) \restriction L$ and $(M^\xi \cap P^{M^\xi}) \restriction L$ are isomorphic, and the same holds for $(M^\eta \backslash P^{M^\eta}) \restriction L$ and $(M^\xi \backslash P^{M^\xi}) \restriction L$. Thus X is invariant under $(L \cup \{P\}, \pi)$ and by Lemma 4.2, there is an $L_{\kappa^+\kappa}$-sentence φ such that $X = \{\eta \in \kappa^\kappa \mid M^\eta \models \varphi\}$. Let α be the rank of φ.

By the way h was defined and our assumptions on $\mathcal{T}$, there are $\eta_0, \eta_1 \in 2^\kappa$ such that $M_{\eta_0} \models \mathcal{T}$, $M_{\eta_1} \models \mathcal{T}$, $M_{\eta_0} \ncong M_{\eta_1}$, and $M_{\eta_0} \equiv^\alpha_{\kappa^+\kappa} M_{\eta_1}$. Let $\eta, \xi \in S$ be such that $h(\eta) = (\eta_0, \eta_0)$ and $h(\xi) = (\eta_0, \eta_1)$. Clearly $\eta \in X$, $\xi \notin X$, and $M_\eta \equiv^\alpha_{\kappa^+\kappa} M_\xi$. This contradicts that φ has rank α.

$\boxminus$

Theorem 4.5. *Suppose $\mathcal{T}$ is a countable complete first order theory. Suppose κ is regular and $\eta \in \kappa^\kappa$ are such that $M_\eta \models \mathcal{T}$, and for all $\alpha < \kappa^+$ there is $\xi \in \kappa^\kappa$ such that $M_\xi \models \mathcal{T}$, $M_\xi \ncong M_\eta$ and $M_\xi \equiv^\alpha_{\kappa^+\kappa} M_\eta$. Then* $\mathrm{Orb}(M_\eta)$ *is not κ-Borel.*

Proof. We will use the same preparation as is Theorem 4.4, i.e. the function h and the models M^η.

Let us assume, for sake of contradiction, that $\mathcal{T}$ and η satisfy the assumptions and $\mathrm{Orb}(M_\eta)$ is κ-Borel. Let

$$\begin{aligned} X &= \{\xi \in S \mid M_{h_1(\xi)} \cong M_{h_2(\xi)} \cong M_\eta\} \\ &= h^{-1}[(\mathrm{Orb}(M_\eta) \times \mathrm{Orb}(M_\eta)) \cap (2^\kappa \times 2^\kappa)], \end{aligned}$$

since $\mathrm{Orb}(M_\eta)$ are X are κ-Borel. Notice that X is invariant under $(L\cup\{P\},\pi)$ and by Lemma 4.2, there is an $L_{\kappa^+\kappa}$-sentence φ such that $X=\{\eta\in\kappa^\kappa\mid M^\eta\models\varphi\}$. Let α be the rank of φ.

By the way h was defined and our assumptions on $\mathcal{T}$ and η, there is $\xi\in 2^\kappa$ such that $M_\xi\models\mathcal{T}$, $M_\eta\not\cong M_\xi$, and $M_\eta\equiv^\alpha_{\kappa^+\kappa} M_\xi$. Let $\zeta,\zeta'\in S$ be such that $h(\zeta)=(\eta,\eta)$ and $h(\zeta)=(\eta,\xi)$. Clearly $\zeta\in X$, $\zeta'\notin X$, and $M_\zeta\equiv^\alpha_{\kappa^+\kappa} M_{\zeta'}$. This contradicts that φ has rank α.

$\boxminus$

Corollary 4.6. *Suppose $\mathcal{T}$ is a countable non-classifiable theory (i.e. $\mathcal{T}$ is not a superstable stable theory without DOP and without OTOP) and $\kappa>\omega$ is regular, then the isomorphism relation of $\mathcal{T}$, $\cong_{\mathcal{T}}$, is not κ-Borel on the Generalized Baire Space κ^κ.*

Proof. From [11] Theorem 0.2 (Main Conclusion), there are 2^κ pairwise non-isomorphic $L_{\infty\kappa}$-equivalent models of power κ. The result follows from Theorem 4.4. $\boxminus$

5. Orbits of uncountable models: the structure case

We prove Borel-definability results for orbits and for the isomorphism relation of models of cardinality κ in the structure case of Shelah's Main Gap. Our assumption is the regularity of κ and $\kappa=\kappa^\omega$. This assumption is, of course, weaker than the assumption $\kappa=\kappa^{<\kappa}$ prevalent in earlier work in this area.

For all $n<\omega$, let $<^n$ be a well-ordering of κ^n of order type κ. For all $n<\omega$ and cardinals θ, let R^n_θ be the set of tuples $a\in\kappa^n$ such that $|\{b\in\kappa^n\mid b<^n a\}|=\theta$. Let $L^+=\{<^n,R^n_\theta\mid n<\omega,\theta<\kappa\}$ and $L^*=L\cup L^+$. We use $L^*_{\kappa^+\lambda}$ to denote the set of sentences of $L_{\kappa^+\lambda}$ when the vocabulary is L^*. We denote by M^*_η the expansion of M_η to L^* with the right interpretation of the elements of L^+ (e.g. $<^n$ is a well-ordering of κ^n of order type κ).

Lemma 5.1. *Let $X\subseteq\kappa^\kappa$. If there is an $L^*_{\kappa^+\lambda}$-sentence φ such that $\eta\in X$ if and only if $M^*_\eta\models\varphi$, whenever $\eta\in\kappa^\kappa$, then X is $(\kappa,\kappa^{<\lambda})$-Borel.*

Proof. Let φ be an $L^*_{\kappa^+\lambda}$-formula with variables $\bar{x}$, let $\gamma<\lambda$ be the arity of $\bar{x}$. It is enough to show that for any free variables assignment $s:\gamma\to\kappa$, the set

$$X^s_\varphi:=\{\eta\in\kappa^\kappa\mid M^*_\eta\models_s\varphi\}$$

is $(\kappa,\kappa^{<\lambda})$-Borel. Let s be a free variables assignment, we will proceed by induction on formulas, ψ, to show that X^s_ψ is $(\kappa,\kappa^{<\lambda})$-Borel.

Case φ is atomic:

- ψ is an L-formula: Let $\psi=R(x_1,x_2,\dots,x_n)$, thus $M^*_\eta\models_s\psi$ if and only if $\eta(\alpha)>0$, where $\alpha=\pi(k,s(1),s(2),\dots,s(n))$ and

$R = Q_k$. Therefore, X^s_ψ is the union of κ basic κ-open sets. So, it is $(\kappa, \kappa^{<\lambda})$-Borel.

- ψ is an L^+-formula: Let $\psi = P(x_1, x_2, \ldots, x_n)$. Notice that for all $N, M \models \psi$, $N \restriction L^+ = M \restriction L^+$. Thus $(s(1), s(2), \ldots, s(n)) \in P^M$ if and only if $(s(1), s(2), \ldots, s(n)) \in P^N$. Therefore $X^s_\psi = \emptyset$ or $X^s_\psi = \kappa^\kappa$.

Case negation: Let $\psi = \neg\psi'$ such that ψ' is a L^*-formula and $X^s_{\psi'}$ is $(\kappa, \kappa^{<\lambda})$-Borel. Since $(\kappa, \kappa^{<\lambda})$-Borel is closed under complement, X^s_ψ is $(\kappa, \kappa^{<\lambda})$-Borel.

Case conjunction and disjunction: Let $\{\psi_i \mid i < \delta < \kappa^{<\lambda}\}$ be a set of L^*-formulas such that for all $X^s_{\psi_i}$ is $(\kappa, \kappa^{<\lambda})$-Borel. Let $\psi = \bigwedge_{i<\delta} \psi_i$, so $X^s_\psi = \bigcap_{i<\delta} X^s_{\psi_i}$. Thus X^s_ψ is $(\kappa, \kappa^{<\lambda})$-Borel. The disjunction case is similar.

Case quantifiers: Let $\{V_i \mid i < \delta < \kappa^{<\lambda}\}$ be a set of variables and ψ' a L^*-formula such that $X^s_{\psi'}$ is $(\kappa, \kappa^{<\lambda})$-Borel. Let $\psi = \exists V_0 \exists V_1 \cdots \psi'$, it is clear that $X^s_\psi = X^s_{\exists \bar{V}\psi'} = \bigcup_{\bar{a}\in\kappa} X^{s(\bar{a}/\bar{V})}_{\psi'}$, thus X^s_ψ is $(\kappa, \kappa^{<\lambda})$-Borel. The universal quantifier case is similar. $\boxminus$

We call $\mathbf{P} \subseteq \mathcal{P}(\kappa^n)$ a property of n-ary relations. We say that a relation $R \subseteq \kappa^n$ has property $\mathbf{P}$ if $R \in \mathbf{P}$. The previous lemma can be generalized to any property $\mathbf{P}$ of n-ary relations, the proof is similar.

Fact 5.2. *To show that a property* $\mathbf{P}$ *of n-ary relations is κ-Borel, it is enough to find L, $\mathcal{A}$, and φ such that.*

(1) *L is a signature;*
(2) *$\mathcal{A}$ is an L-structure with domain κ;*
(3) *$\varphi \in L_{\kappa^+\omega}$ with the signature $L \cup \{R\}$*
(4) *$\forall R \subseteq \kappa^2$, R has the property* $\mathbf{P}$ *if and only if $(\mathcal{A}, R) \models \varphi$.*

The following surprising lemma is in sharp contrast to the famous result of classical Descriptive Set Theory to the effect that the set of elements of ω^ω that code a well-order is Π^1_1-complete and therefore not Borel.

Lemma 5.3. *If* $\mathrm{cf}(\kappa) > \omega$*, then the property of $R \subseteq \kappa \times \kappa$ being a well-ordering is κ-Borel.*

Proof. We use Fact 5.2. Let $L = \{(S_\alpha)_{\alpha<\kappa}\}$, $\mathcal{A} = (\kappa, (S_\alpha)_{\alpha<\kappa})$, where the interpretation of S_α is $\{\beta < \kappa \mid \beta < \alpha\}$. Let R be a relational symbol interpretd as a linear order. For all $\alpha, \beta < \kappa$ and $i < \omega$, we define $\varphi^\alpha_\beta(x_i)$ in a recursive way as follows:

- $\varphi^\alpha_0(x_i) :=$ "$x_i = x_i$",
- if $\beta > 0$, then $\varphi^\alpha_\beta(x_i) := \bigwedge_{\gamma_{i+1}<\beta} \exists x_{i+1}(S_\alpha(x_{i+1}) \wedge (x_{i+1}\ R\ x_i) \wedge \varphi^\alpha_{\gamma_{i+1}}(x_{i+1}))$.

Let $\varphi := \bigvee_{\alpha<\kappa} \bigwedge_{\gamma_0<\kappa} \exists x_0(S_\alpha(x_0) \wedge \varphi^\alpha_{\gamma_0}(x_0))$. Notice that (κ, R) is a well-order if and only if for all $\alpha < \kappa$, $(\alpha, R \cap S^2_\alpha)$ is a well-order.

Claim 5.3.1. *(κ, R) is not a well-order if and only if $(\kappa, R, (S_\alpha)_{\alpha<\kappa}) \models \varphi$.*

Proof. $\Rightarrow$) Suppose (κ, R) is not a well-order. Then there is $\alpha < \kappa$ such that $(\alpha, R \cap S_\alpha^2)$ is not a well-order. So there is a sequence $\langle a_i \mid i < \omega\rangle$ such that for all $i < \omega$, $a_{i+1} R a_i$. Let us show that for all $\gamma_0 < \kappa$, II has a winning strategy for the semantic game for

$$(\kappa, R, (S_\beta)_{\beta<\kappa}) \models \exists x_0 (S_\alpha(x_0) \wedge \varphi^\alpha_{\gamma_0}(x_0)).$$

In the ith-round, II chooses a_i as an interpretation for x_i. Since $\langle a_i \mid i < \omega\rangle$ is an infinite descending sequence, a_i is a valid move and II doesn't lose. Thus $(\kappa, R, (S_\beta)_{\alpha<\kappa}) \models \varphi$.

$\Leftarrow$) Suppose that (κ, R) is a well-order. Let us show that for all $\alpha < \kappa$, I has a winning strategy for the semantic game for

$$(\kappa, R, (S_\beta)_{\beta<\kappa}) \models \bigwedge_{\gamma_0<\kappa} \exists x_0 (S_\alpha(x_0) \wedge \varphi^\alpha_{\gamma_0}(x_0)).$$

Let us describe a winning strategy for I. I start by choosing $\gamma_0 = otp(S_\alpha, R \cap S_\alpha^2)$. Suppose that in the ith-round, I has chosen $\langle \gamma_j \mid j \leq i\rangle$, II has chosen $\langle a_j \mid j \leq i\rangle$, and II has not lost. Then I chooses $\gamma_i = otp(A_i, R \cap A_i^2)$, where $A_i = \{\delta \in S_\alpha \mid \delta R a_i\}$. To show that this is a winning strategy, notice that $(\alpha, R \cap S_\alpha^2)$ is a well-order (since R is a well-order). Thus, no matter how II plays, there is $j < \omega$ such that in the ith-round II has chosen the R-least element a_j of α. So, $\gamma_j = \emptyset$ and II cannot choose an interpretation of x_{j+1} without losing. Thus $(\kappa, R, (S_\alpha)_{\alpha<\kappa}) \not\models \varphi$. ⊟

⊟

Theorem 5.4. *Let $\mathcal{T}$ be a countable ω-stable NDOP shallow L-theory. Suppose κ is a regular cardinal such that $\kappa^\omega = \kappa$, and let $\eta \in \kappa^\kappa$ be such that $M_\eta \models \mathcal{T}$. Then there is $\varphi \in L^*_{\kappa^+\omega_1}$ such that $M_\xi \cong M_\eta$ if and only if $M^*_\xi \models \varphi$. Hence* Orb(M_η) *is κ-Borel.*

Let us make some preparations before we prove Theorem 5.4. From now on in this section, we will work under the assumption that $\mathcal{T}$ is a countable ω-stable NDOP shallow L-theory. Let us fix η and $\mathcal{T}$ such that satisfy the assumptions of Theorem 5.4, thus $M_\eta \models \mathcal{T}$.

Suppose $\mathcal{B}$ and $\mathcal{A}$ are countable such that $\mathcal{B} \preceq \mathcal{A} \preceq M$, where M is a model of $\mathcal{T}$ (not necessary M_η). We will write $tp(a, \mathcal{A}) \dashv \mathcal{B}$ for models $\mathcal{A}$ and $\mathcal{B}$, when $tp(a, \mathcal{A})$ is orthogonal to $\mathcal{B}$. We will denote by $\mathcal{B}[a]$ the primary model over $\mathcal{B}a$.

Given a tree T, for all $t \in T$ we define $ht_T(t)$ as the order type of $\{u \in T \mid u < t\}$. We say that $T' \subseteq T$ is a subtree of T if T' is a substructure of T, T' has a root, and for all $t < t' < t''$ in T, if $t, t'' \in T'$, then $t' \in T'$.

Recall the rank of a tree T without infinite branches, $rk(T)$ from Definition 3.9.

Definition 5.5. We say that $(T, a_t, \mathcal{A}_t)_{t\in T}$ is a good labelled tree of M (we drop M when it is clear from the context) if the following holds:

(1) T is a subtree of $(\kappa^+)^{<\omega}$.
(2) If $t^\frown i \in T$, and $j < i$, then $t^\frown j \in T$.
(3) If $ht_T(t) = 0$, then $\mathcal{A}_t \prec M$ is primary over the empty set and $a_t \in \mathcal{A}_t$.
(4) If $t \in T$ and $ht_T(t) \geq 1$, then $\mathcal{A}_t = \mathcal{A}_{t^-}[a_t] \preceq M$ and $a_t \not\subseteq \mathcal{A}_{t^-}$.
(5) If $t \in T$ and $ht_T(t) \geq 2$, then $tp(a_t/\mathcal{A}_{t^-}) \dashv \mathcal{A}_{t^{--}}$.
(6) If $t^\frown i \in T$, then

$$a_{t^\frown i} \downarrow_{\mathcal{A}_t} \bigcup_{j<i} a_{t^\frown j}.$$

We say that $(T, a_t, \mathcal{A}_t)_{t\in T}$ is a good labelled tree over a countable $\mathcal{A} \prec M$ of M, if $(T, a_t, \mathcal{A}_t)_{t\in T}$ satisfies (1), (2), (4), (5), and (6), and the following:

(3′) If $ht_T(t) = 0$, then $\mathcal{A}_t = \mathcal{A}$ and $a_t \in \mathcal{A}$.

Definition 5.6. We say that $(T, a_t, \mathcal{A}_t)_{t\in T}$ is a good labelled tree over $(\mathcal{B}, \mathcal{A})$, $\mathcal{B} \prec \mathcal{A} \prec M$ countable, of M if the following holds:

(1) T is a subtree of $(\kappa^+)^{<\omega}$.
(2) If $t^\frown i \in T$, $ht_T(t) \geq 1$, and $j < i$, then $t^\frown j \in T$.
(3) If $ht_T(t) = 0$, then $\mathcal{A}_t = \mathcal{B}$, t has a unique immediate successor, t^+, $\mathcal{A}_{t^+} = \mathcal{A}$, $a_t \in \mathcal{B}$ and $a_{t^+} \in \mathcal{A}$.
(4) If $t \in T$ and $ht_T(t) \geq 1$, then $\mathcal{A}_t = \mathcal{A}_{t^-}[a_t] \preceq M$ and $a_t \not\subseteq \mathcal{A}_{t^-}$.
(5) If $t \in T$ and $ht_T(t) \geq 2$, then $tp(a_t/\mathcal{A}_{t^-}) \dashv \mathcal{A}_{t^{--}}$.
(6) If $t^\frown i \in T$, then

$$a_{t^\frown i} \downarrow_{\mathcal{A}_t} \bigcup_{j<i} a_{t^\frown j}.$$

Notice that since $\mathcal{A}_t = \mathcal{A}_{t^-}[a_t]$, $a_t \triangleright_{\mathcal{A}_{t^-}} \mathcal{A}_t$.

Definition 5.7. A good labelled tree $(T, a_t, \mathcal{A}_t)_{t\in T}$ is maximal if for all $t \in T$, there is no $a \in M$ such that $a \downarrow_{\mathcal{A}_t} \bigcup\{\mathcal{A}_{t^\frown i} \mid t^\frown i \in T\}$, $tp(a/\mathcal{A}_t)$ is non-algebraic, and in case $ht_T(t) > 0$, $tp(a/\mathcal{A}_t) \dashv \mathcal{A}_{t^-}$ (i.e. there is no proper extension of the good labelled tree). A maximal good labelled tree over $\mathcal{A}$, or over $(\mathcal{B}, \mathcal{A})$, is defined in a similar way, following the definition above.

Definition 5.8. Suppose $(T, a_t, \mathcal{A}_t)_{t\in T}$ is a good labelled tree and $(T', a'_t, \mathcal{A}'_t)_{t\in T'}$ is a good labelled tree. Then (f, g) is an isomorphism from $(T, a_t, \mathcal{A}_t)_{t\in T}$ to $(T', a'_t, \mathcal{A}'_t)_{t\in T'}$ if f is an isomorphism from $(T, <)$ to $(T', <')$ and $g : \cup_{t\in T}\mathcal{A}_t \to \cup_{t\in T'}\mathcal{A}'_t$ is such that for all $t \in T$, $g \restriction \mathcal{A}_t$ is an isomorphism from $\mathcal{A}_t$ to $\mathcal{A}'_{f(t)}$ and for all $t \in T$, $ht_T(t) \geq 1$, $g(a_t) = a'_{f(t)}$.

If $(T, a_t, \mathcal{A}_t)_{t\in T}$ and $(T', a'_t, \mathcal{A}'_t)_{t\in T'}$ are good labelled trees over $\mathcal{A}_r$, r the root of T and T' (T and T' have the same root). Then we say that they are isomorphic over $\mathcal{A}_r$ if there is an isomorphism (f,g) such that $g \restriction \mathcal{A}_r = id_{\mathcal{A}_r}$.

Fact 5.9. *If $b \in \mathcal{A}[a]$ and $tp(a/\mathcal{A}) \dashv \mathcal{B}$, then $tp(b/\mathcal{A}) \dashv \mathcal{B}$.*

Proof. Let us assume, for sake of contradiction, that $b \in \mathcal{A}[a]$, $tp(a/\mathcal{A}) \dashv \mathcal{B}$, and $tp(b/\mathcal{A})$ is not orthogonal to $\mathcal{B}$. Therefore, there are c and d such that $c \downarrow_{\mathcal{A}} b$, $d \downarrow_{\mathcal{B}} \mathcal{A}c$, and $b \not\downarrow_{\mathcal{A}c} d$. Without loss of generality, we can choose c such that $c \downarrow_{\mathcal{A}b} a$. Since $c \downarrow_{\mathcal{A}} b$, by transitivity we conclude $c \downarrow_{\mathcal{A}} a$. On the other hand $tp(a/\mathcal{A}) \dashv \mathcal{B}$ and $d \downarrow_{\mathcal{B}} \mathcal{A}c$, thus $a \downarrow_{\mathcal{A}c} d$. Therefore, $a \downarrow_{\mathcal{A}} cd$. Since $a \rhd_{\mathcal{A}} b$, $b \downarrow_{\mathcal{A}} cd$. This implies $b \downarrow_{\mathcal{A}c} d$, a contradiction. $\boxminus$

For all $t \in T$ we will denote by $T_{\geq t}$ the subtree $\{a \in T \mid a \geq t\}$. Let $t \in T$,

- If t^- exists, then we define T_t as the subtree $T_{\geq t} \cup \{t^-\}$;
- If t^- does not exist, then we define T_t as the subtree $T_{\geq t}$.

Definition 5.10. We say that a good labelled tree $(T, a_t, \mathcal{A}_t)_{t\in T}$ of M is a QD (quasi-decomposition) of M (we drop M when it is clear from the context) if the following holds:

(1) If $t^\frown i \in T$, then $MR(tp(a_{t^\frown i}/\mathcal{A}_t))$ is the least possible, i.e. there is no $a \in M$ such that it satisfies that $a \not\subseteq \mathcal{A}_t$, Definition 5.5 (5) and (6), and $MR(tp(a/\mathcal{A}_t)) < MR(tp(a_{t^\frown i}/\mathcal{A}_t))$. For all $t \in T$ such that $ht_T(a_t) \geq 1$, we write p_t for $tp(a_t/\mathcal{A}_{t^-})$.

(2) If $t^\frown i \in T$, $k < j < i$, and $p_{t^\frown k} = p_{t^\frown i}$, then $p_{t^\frown j} = p_{t^\frown i}$.
If $t^\frown i \in T$ and for all $j < i$, $p_{t^\frown i} \neq p_{t^\frown j}$, then for all $j < i$, there is no $a \in M$ such that $tp(a/\mathcal{A}_t) = p_j$ and
$$a \downarrow_{\mathcal{A}_t} \bigcup_{\beta<i} a_{t^\frown \beta}.$$

(3) If $t^\frown i \in T$, $k < j < i$, and $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown i}}$ is a maximal good labelled tree over $(\mathcal{A}_t, \mathcal{A}_{t^\frown i})$ and $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown i}}$ is isomorphic to $(T_{t^\frown k}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown k}}$ over $\mathcal{A}_t$, then $(T_{t^\frown j}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown j}}$ is isomorphic to $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown i}}$ over $\mathcal{A}_t$.
If $t^\frown i \in T$, is such that $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown i}}$ is a maximal good labelled tree over $(\mathcal{A}_t, \mathcal{A}_{t^\frown i})$ and for all $j < i$, $(T_{t^\frown j}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown j}}$ is not isomorphic to $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown i}}$ over $\mathcal{A}_t$. Then for all $j < i$, there is no good labelled tree $(T', a'_u, \mathcal{A}'_u)_{u\in T'}$ over $(\mathcal{A}'_r, \mathcal{A}'_{r^+})$, where r is the root of T', such that it is a maximal labelled tree over $(\mathcal{A}'_r, \mathcal{A}'_{r^+})$, $\mathcal{A}'_r = \mathcal{A}_t$, and $(T', a'_u, \mathcal{A}'_u)_{u\in T'}$ is isomorphic to $(T_{t^\frown j}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown j}}$ over $\mathcal{A}_t$ and
$$a_{r^+} \downarrow_{\mathcal{A}_t} \bigcup_{k<i} a_{t^\frown k}.$$

(4) If $t^\frown i \in T$, $j < i$, then $(T_{t^\frown j}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown j}}$ is a maximal labelled tree over $(\mathcal{A}_t, \mathcal{A}_{t^\frown j})$.

We define a QD over $\mathcal{A}$ and QD over $(\mathcal{B}, \mathcal{A})$, in a similar way.

We denote by Y^i_t the set of those $j < \kappa^+$ such that $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u \in T_{t^\frown i}}$ is isomorphic to $(T_{t^\frown j}, a_u, \mathcal{A}_u)_{u \in T_{t^\frown j}}$ over $\mathcal{A}_t$. Notice that the Y^i_t's are intervals of κ^+.

Definition 5.11. A QD $(T, a_t, \mathcal{A}_t)_{t \in T}$ is maximal if $(T, a_t, \mathcal{A}_t)_{t \in T}$ is a maximal good labelled tree. In a similar way, we define maximal QD over $\mathcal{A}$ and maximal QD over $(\mathcal{B}, \mathcal{A})$.

Lemma 5.12. *M_η has a maximal QD.*

Proof. We will construct a maximal QD by recursion. Let us start by constructing $(T^0, \mathcal{A}^0_t)_{t \in T^0}$. Let $T^0 = \{\emptyset\}$, $\mathcal{A}^0_\emptyset \subseteq M_\eta$ a primary model over $\emptyset$, and $a^0_\emptyset \in \mathcal{A}^0_\emptyset$.

Suppose that $\alpha < \kappa^+$ is such that $(T^\alpha, a^\alpha_t, \mathcal{A}^\alpha_t)_{t \in T^\alpha}$ has been constructed. If $(T^\alpha, a^\alpha_t, \mathcal{A}^\alpha_t)_{t \in T^\alpha}$ is a maximal QD, then we are done. Let us take care of the case when $(T^\alpha, a^\alpha_t, \mathcal{A}^\alpha_t)_{t \in T^\alpha}$ is not a maximal QD. Since $(T^\alpha, a^\alpha_t, \mathcal{A}^\alpha_t)_{t \in T^\alpha}$ is not a maximal QD, there is $t \in T^\alpha$ such that $ht_{T^\alpha}(t) > 0$ and $(T^\alpha_t, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha_t}$ is not a maximal labelled tree over $(\mathcal{A}^\alpha_{t^-}, \mathcal{A}^\alpha_t)$, or $ht_{T^\alpha}(t) = 0$ and $(T^\alpha_t, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha_t}$ is not a maximal labelled tree.

Let us pick $t \in T^\alpha$ such that $ht_{T^\alpha}(t)$ is maximal with such property. Therefore, for all $i < \kappa^+$, if $t^\frown i \in T^\alpha$, then $(T^\alpha_{t^\frown i}, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha_{t^\frown i}}$ is a maximal labelled tree over $(\mathcal{A}^\alpha_t, \mathcal{A}^\alpha_{t^\frown i})$. Let $k = \min\{i < \kappa^+ \mid t^\frown i \notin T^\alpha\}$. By Definition 5.10 we will have to deal with three cases.

(1) There are $j < k$ and $(T', a_u, \mathcal{A}_u)_{u \in T'}$ such that for all $k > i > j$, $(T^\alpha_{t^\frown i}, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha_{t^\frown i}}$ is isomorphic to $(T^\alpha_{t^\frown j}, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha_{t^\frown j}}$ over $\mathcal{A}^\alpha_t$, and $(T', a_u, \mathcal{A}_u)_{u \in T'}$ is isomorphic to $(T^\alpha_{t^\frown j}, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha_{t^\frown j}}$ over $\mathcal{A}^\alpha_t$, and it is a maximal good labelled tree over $(\mathcal{A}^\alpha_t, \mathcal{A}_{u^*})$ which is also a QD, where u^* is such that $ht_{T'}(u^*) = 1$ and
$$a_{u^*} \downarrow_{\mathcal{A}_t} \bigcup_{\beta < k} a^\alpha_{t^\frown \beta}.$$
We define $(T^{\alpha+1}_{t^\frown k}, a^{\alpha+1}_u, \mathcal{A}^{\alpha+1}_u)_{u \in T^{\alpha+1}_{t^\frown k}}$ as $(T', a_u, \mathcal{A}_u)_{u \in T'}$ and
$$(T^{\alpha+1}, a^{\alpha+1}_u, \mathcal{A}^{\alpha+1}_u)_{u \in T^{\alpha+1}}$$
as
$$(T^\alpha, a^\alpha_u, \mathcal{A}^\alpha_u)_{u \in T^\alpha} \cup (T^{\alpha+1}_{\geq t^\frown k}, a^{\alpha+1}_u, \mathcal{A}^{\alpha+1}_u)_{u \in T^{\alpha+1}_{\geq t^\frown k}}.$$

(2) There are no $j < k$ and $(T', a_u, \mathcal{A}_u)_{u \in T'}$ as in the previous case, but there are $j < k$ and $a \not\subseteq \mathcal{A}^\alpha_t$ such that for all $k > i > j$, $p_{t^\frown j} = p_{t^\frown i}$ and $tp(a/\mathcal{A}^\alpha_t) = p_{t^\frown j}$ and
$$a \downarrow_{\mathcal{A}^\alpha_t} \bigcup_{\beta < k} a^\alpha_{t^\frown \beta}.$$
We define $a^{\alpha+1}_{t^\frown k} = a$, $\mathcal{A}^{\alpha+1}_{t^\frown k} = \mathcal{A}^\alpha_t[a]$, and $T^{\alpha+1} = T^\alpha \cup \{t^\frown k\}$.

(3) There are no $j<k$ and $(T',a_u,\mathcal{A}_u)_{u\in T'}$ as in the first case, neither $j<k$ and $a\not\subseteq\mathcal{A}_t^\alpha$ as in the second case.
   (a) Case $ht_{T^\alpha}(t)>0$.
   We choose $a\not\subseteq\mathcal{A}_t^\alpha$ such that $tp(a/\mathcal{A}_t^\alpha)\dashv\mathcal{A}_{t^-}^\alpha$, $tp(a/\mathcal{A}_t^\alpha)$ is non-algebraic,
$$a\downarrow_{\mathcal{A}_t^\alpha}\bigcup_{\beta<k}a_{t\frown\beta}^\alpha,$$
   and with the least possible $MR(tp(a/\mathcal{A}_t^\alpha))$. We define $a_{t\frown k}^{\alpha+1}=a$, $\mathcal{A}_{t\frown k}^{\alpha+1}=\mathcal{A}_t^\alpha[a]$, and $T^{\alpha+1}=T^\alpha\cup\{t\frown k\}$.
   (b) Case $ht_{T^\alpha}(t)=0$.
   We choose $a\not\subseteq\mathcal{A}_t^\alpha$ such that $tp(a/\mathcal{A}_t^\alpha)$ is non-algebraic,
$$a\downarrow_{\mathcal{A}_t^\alpha}\bigcup_{\beta<k}a_{t\frown\beta}^\alpha,$$
   and with the least possible $MR(tp(a/\mathcal{A}_t^\alpha))$. We define $a_{t\frown k}^{\alpha+1}=a$, $\mathcal{A}_{t\frown k}^{\alpha+1}=\mathcal{A}_t^\alpha[a]$, and $T^{\alpha+1}=T^\alpha\cup\{t\frown k\}$.

Suppose α is a limit ordinal such that for all $\gamma<\alpha$, $(T^\gamma,a_t^\gamma,\mathcal{A}_t^\gamma)_{t\in T^\gamma}$ has been constructed. We define $(T^\alpha,a_t^\alpha,\mathcal{A}_t^\alpha)_{t\in T^\alpha}$ by

- $T^\alpha=\bigcup_{\gamma<\alpha}T^\gamma$.
- For all $t\in T^\alpha$, $a_t^\alpha=a_t^\gamma$ where γ is the least ordinal such that $t\in T^\gamma$.
- For all $t\in T^\alpha$, $\mathcal{A}_t^\alpha=\mathcal{A}_t^\gamma$ where γ is the least ordinal such that $t\in T^\gamma$.

$\boxminus$

Let $(T,a_t,\mathcal{A}_t)_{t\in T}$ be the maximal QD of M_η from Lemma 5.12.

Lemma 5.13. *If $t\in T$, $j\le i$, a, and b are such that $a\models p_{t\frown i}$, $b\models p_{t\frown j}$, $\mathcal{A}_t\subseteq A$, $a\downarrow_{\mathcal{A}_t}A$, and $b\not\downarrow_{\mathcal{A}_t}A$. Then $a\downarrow_A b$. In particular $p_{t\frown i}$ is regular.*

Proof. Let us assume, for sake of contradiction, that $t\in T$, $j\le i$, a, and b are such that $a\models p_{t\frown i}$, $b\models p_{t\frown j}$, $\mathcal{A}_t\subseteq A$, $a\downarrow_{\mathcal{A}_t}A$, and $b\not\downarrow_{\mathcal{A}_t}A$, and $a\not\downarrow_A b$. So there is $d\in A$ such that $a\downarrow_{\mathcal{A}_t}d$, and $b\not\downarrow_{\mathcal{A}_t}d$, and $a\not\downarrow_{\mathcal{A}_td}b$.

Let us choose $c\in\mathcal{A}_t$ such that $a\downarrow_c\mathcal{A}_t$, $b\downarrow_c\mathcal{A}_t$, $b\not\downarrow_c d$, and there is $\varphi(y,c)\in tp(b,\mathcal{A}_t)$ such that $MR(\varphi(y,c))=MR(tp(b,\mathcal{A}_t))$ and has degree 1, and $a\not\downarrow_c db$.

Notice that there is $\psi(x,y,a,c)$ such that $\models\psi(d,b,a,c)$ and $\psi(x,y,a,c)$ forks over c. Also, there is $\chi(y,d,c)$ such that $\models\chi(b,d,c)$ and $\chi(b,d,c)$ forks over c. Without loss of generality, we may assume that there is $d_\chi(x,c)$ that defines $tp(b,\mathcal{A}_t)\restriction\chi(y,x,c)$. Let us define
$$\theta'(x,y,z,c)=\varphi(y,c)\wedge\psi(x,y,z,c)\wedge\neg d_\chi(x,c)\wedge\chi(y,x,c)$$
and $\theta(x,z,c)=\exists y\theta'(x,y,z,c)$. Notice that $\models\theta(d,a,c)$ and $a\downarrow_c d$, therefore there is $d'\in\mathcal{A}_t$ such that $\models\theta(d',a,c)$. Thus there is $b'\in$

$\mathcal{A}_{t^\frown i}$ such that $\models \theta'(d', b', a, c)$, so $\models \psi(d', b', a, c)$. Since $a \not\downarrow_c d'b'$, $b' \in \mathcal{A}_t[a_{t^\frown i}]\backslash\mathcal{A}_t$ and $a \triangleright_{\mathcal{A}_t} b'$. We conclude that $b' \downarrow_{\mathcal{A}_t} \bigcup_{j<i} a_{t^\frown j}$, $tp(b', \mathcal{A}_t)$ is not algebraic, and $tp(b', \mathcal{A}_t) \dashv \mathcal{A}_{t^-}$.

Since $\not\models d_\chi(d', c)$, $\chi(y, d', c) \notin tp(b, \mathcal{A}_t)$. On the other hand $\models \chi(b', d', c)$, so $tp(b', \mathcal{A}_t) \neq tp(b, \mathcal{A}_t)$. Since $MR(\varphi(y, c)) = MR(tp(b, \mathcal{A}_t))$ and has degree 1, $MR(tp(b', \mathcal{A}_t)) < MR(tp(b, \mathcal{A}_t))$ a contradiction with the minimality of $MR(tp(b, \mathcal{A}_t))$.

$\boxminus$

Corollary 5.14. *Maximal QD are decompositions. In particular, M_η and M_ξ have isomorphic QD's if and only if $M_\eta \cong M_\xi$.*

Proof. The first claim follows from the *"in particular"* part of Lemma 5.13. For the second claim, clearly $M_\eta \cong M_\xi$ implies that M_η and M_ξ have isomorphic QD's. For the other direction, from the proof of Lemma 5.13, it can be seen that $tp(a_t, \mathcal{A}_{t^-})$ are weakly isolated, thus strongly regular (see [1] page 243). Therefore a maximal QD is a K-representation as in Definition 2.1 [1] page 347. From Corollary 2.4 [1] page 348, M_η and M_ξ are K-minimal over their QD's. We conclude that if M_η and M_ξ have isomorphic QD's, then $M_\eta \cong M_\xi$.

$\boxminus$

We will find an $L^*_{\kappa^+\omega_1}$-formula Φ_η such that $M^*_\xi \models \Phi_\eta$ if and only if $M_\xi \cong M_\eta$, i.e. there is a maximal QD of M_ξ, which is isomorphic to the maximal one found in Lemma 5.12.

We prove by induction on $rk(t)$ for $t \in T$, $ht_T(t) > 0$, the following:

There is an $L^*_{\kappa^+\omega_1}$-formula $\Phi^t_\eta(\bar{X}, \bar{Y}, x)$ such that for all ξ, countable $\mathcal{B} \prec \mathcal{A} \prec M_\xi$ and $a \in \mathcal{A}$; $M_\xi \models \Phi^t_\eta(\mathcal{A}, \mathcal{B}, a)$ if and only if there is a maximal QD, $(T', a'_u, \mathcal{A}'_u)_{u\in T'}$, over $(\mathcal{B}, \mathcal{A})$ such that $a'_{u^*} = a$ where $ht_{T'}(u^*) = 1$ and $(T', a'_u, \mathcal{A}'_u)_{u\in T'}$ is isomorphic with $(T_t, a_u, \mathcal{A}_u)_{u\in T_t}$ over $\mathcal{A}_t$.

Case $rk(t) = 0$, i.e. a leaf. Let $\Phi^t_\eta(\bar{X}, \bar{Y}, y)$ be the conjunction of all first order formulas $\varphi(\bar{X}, \bar{Y}, y)$ such that $M_\eta \models \varphi(\mathcal{A}_t, \mathcal{A}_{t^-}, a_t)$ together with a formula that says that there is no a such that $tp(a, \bar{X})$ is non-algebraic and orthogonal to $\bar{Y}$.

Case $rk(t) > 0$ and $ht_T(t) > 0$. Let $\langle Y_\alpha\rangle_{\alpha<\gamma}$ be an increasing enumeration of the set $\{Y^i_t \mid i < \kappa^+\}$, i.e. if $i < j$, $i \in Y_\alpha$ and $j \in Y_\beta$, then $\alpha \leq \beta$. Notice that there are κ-many finite tuples in M_η, thus $\gamma < \kappa^+$. By the induction assumption, for all $\alpha < \gamma$ and $i \in Y_\alpha$, let $\Phi^\alpha_\eta = \Phi^{t^\frown i}_\eta$ and notice that since for all $i, j \in Y_\alpha$, $(T_{t^\frown i}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown i}}$ and $(T_{t^\frown j}, a_u, \mathcal{A}_u)_{u\in T_{t^\frown j}}$ are isomorphic over $\mathcal{A}_t$, we can construct the formulas $\Phi^{t^\frown i}_\eta$ so that it does not depends on $i \in Y_\alpha$.

We order $\gamma \times \kappa$ lexicographically, i.e. $(\alpha_0, i_0) < (\alpha_1, i_1)$ if either $\alpha_0 < \alpha_1$, or $\alpha_0 = \alpha_1$ and $i_0 < i_1$. For all $\alpha < \gamma$ and $i < \kappa$ we define $\Psi^i_\alpha(\bar{X}, x)$:

x is the $<^n$-least tuple such that:

- *$\exists\bar{Z}\Phi^{\alpha}_{\eta}(\bar{Z},\bar{X},x)$ holds and so*
- *for all $(\alpha_0,i_0)<(\alpha_1,i_1)<\cdots<(\alpha_k,i_k)<(\alpha,i)$, if $x_0,\dots,x_k$ are such that for all $j<k$, x_j satisfies $\Psi^{i_j}_{\alpha_j}$, then $x\downarrow_{\bar{X}} x_0,\dots,x_k$.*

Notice that not necessarily each Ψ^i_α's has a realization and in case it has, it is unique. For all $\alpha<\gamma$, let us denote by $\gamma_\alpha<\kappa$ the least i such that Ψ^i_α's has no realization (in case it exists). Notice that if $\gamma_\alpha\le j$, then Ψ^j_α's has no realization.

For all $\alpha<\gamma$, let us define Ψ_α as follows:

- $\bigwedge_{i<\kappa}\exists x\Psi^i_\alpha$ if γ_α does not exist.
- $\bigwedge_{i<|\gamma_\alpha|}\exists x\Psi^i_\alpha\ \wedge\ \bigvee_{i<|\gamma_\alpha|^+}\neg\exists x\Psi^i_\alpha$ if γ_α exists.

Let $\Phi^t_\eta(\bar{X},\bar{Y},y)$ be the formula that says:

- $\bigwedge_{\alpha<\gamma}\Psi_\alpha$,
- conjunction of all first order formulas $\varphi(\bar{X},\bar{Y},y)$ such that $\models\varphi(\mathcal{A}_t,\mathcal{A}_{t^-},a_t)$,
- there is no tuple a such that $tp(a/\bar{X})$ is non-algebraic, $tp(a/\bar{X})\dashv\bar{Y}$ and if $x_0,\dots,x_n$ satisfies the formulas Ψ^i_α, then $a\downarrow_{\bar{X}}x_0,\dots,x_k$.

Notice that for t such that $ht_T(t)=0$ we do not need to talk about $\bar{Y}$ nor y. For t such that $ht_T(t)=0$, we define γ as above and for all $\alpha<\gamma$, we define Ψ_α as above. Thus $\Phi^t_\eta(\bar{X})=\bigwedge_{\alpha<\gamma}\Psi_\alpha$.

Finally $\Phi_\eta=\exists\bar{X}\Phi^t_\eta(\bar{X})$ for t the root of T.

Now we can proof Theorem 5.4 by showing that Φ_η is the desire formula.

Proof. [Proof Theorem 5.4] $\Leftarrow$) It is clear that if $M^*_\xi\models\Phi_\eta$ and $M^*_{\xi'}\models\Phi_\eta$, then M_ξ and $M_{\xi'}$ have isomorphic QD's and thus $M_\xi\cong M_{\xi'}$.

$\Rightarrow$) We will start by showing the following

- $M^*_\eta\models\Phi_\eta$,
- if $M^*_\eta\models\Phi_\eta$ and $M_\xi\cong M_\eta$, then $M^*_\xi\models\Phi_\eta$.

For this we will show first that $M^*_\eta\models\Phi^t_\eta(\mathcal{A}_t)$ where t is the root of T. For this we will show first that for all $t\in T$ such that $ht_T(t)>0$, $M^*_\eta\models\Phi^t_\eta(\mathcal{A}_t,\mathcal{A}_{t^-},a_t)$ holds. We will proceed by induction over $rk(t)$.

Case $rk(t)=0$. $M^*_\eta\models\Phi^t_\eta(\mathcal{A}_t,\mathcal{A}_{t^-},a_t)$ follows from the construction of $\Phi^t_\eta(\mathcal{A}_t,\mathcal{A}_{t^-},a_t)$ and the fact that $(T_t,a_u,\mathcal{A}_u)_{u\in T_t}$ is a maximal QD of M_η.

Case $rk(t)>0$. As in the construction of Φ^t_η, let Y_α, $\alpha<\gamma$, be the enumeration of the sets Y^i_t. For all $\alpha<\gamma$, let Y'_α be the set of those $i<\kappa$ such that $\psi^i_\alpha(\mathcal{A}_t,x)$ has a realization. We call this realization as b^i_α. We will prove that for all $\alpha<\gamma$, the following hold:

(1) $|Y'_\alpha|=|Y_\alpha|$;
(2) for all $i\in Y'_\alpha$, $b^i_\alpha\not\downarrow_{\mathcal{A}_t}\bigcup_{\beta\le\alpha}\bigcup_{j\in Y_\beta}a_{t^\frown j}$;
(3) for all $i\in Y_\alpha$, $a_{t^\frown i}\not\downarrow_{\mathcal{A}_t}\bigcup_{\beta\le\alpha}\bigcup_{j\in Y_\beta}b^j_\beta$.

Let α be such that for all $\beta<\alpha$, (1), (2), and (3) hold. Let W be the set of all $a\in M^*_\eta$ such that it satisfies Φ^α_η and $a\downarrow_{\mathcal{A}_t}\bigcup_{\beta<\alpha}\bigcup_{j\in Y_\beta}a_{t^\frown j}$, and W' be the set of all $a \in M_\eta^*$ such that it satisfies Φ_η^α and $a \downarrow_{\mathcal{A}_t} \bigcup_{\beta<\alpha} \bigcup_{j \in Y_\beta} b_\beta^j$. Notice that $\{a_t^i \mid i \in Y_\alpha\}$ is a basis of W and $\{b_\alpha^i \mid i \in Y'_\alpha\}$ is a basis of W'. So it is enough to show that $W = W'$. We will show that $W \subseteq W'$, the other direction is is similar.

We will proceed by contradiction, to show that for all $a \in W$, $a \downarrow_{\mathcal{A}_t} \bigcup_{\beta<\alpha} \bigcup_{j \in Y_\beta} b_\beta^j$ (i.e. $W \subseteq W'$). Let us assume, for sake of contradiction, that there are $a \in W$, $n < \omega$, and $b_0, \dots, b_n \in \{b_\beta^i \mid i \in Y'_\beta, \beta < \alpha\}$ such that $a \not\downarrow_{\mathcal{A}_t} b_0 \cdots b_n$. Since $a \downarrow_{\mathcal{A}_t} \bigcup_{\beta<\alpha} \bigcup_{j \in Y_\beta} a_{t^\frown j}$,

$$(*)\ a \not\downarrow_{\mathcal{A}_t \bigcup_{\beta<\alpha} \bigcup_{j \in Y_\beta} a_{t^\frown j}} b_0 \cdots b_n.$$

We choose n so that it is the least for which $(*)$ holds. Let us denote $\bigcup_{\beta<\alpha} \bigcup_{j \in Y_\beta} a_{t^\frown j}$ by Z^*. It is enough to prove the claim below, it contradicts $(*)$.

Claim 5.14.1. $a \downarrow_{\mathcal{A}_t Z^*} b_0 \cdots b_n$.

Proof. Since by the induction assumption $b_n \not\downarrow_{\mathcal{A}_t} Z^*$ and by the choice of n, $a \downarrow_{\mathcal{A}_t Z^*} b_0 \cdots b_{n-1}$, by Lemma 5.13, $tp(a, \mathcal{A}_t Z^* b_0 \cdots b_{n-1})$ and $tp(b_n, \mathcal{A}_t Z^* b_0 \cdots b_{n-1})$ are orthogonal. Since

$$b_n \downarrow_{\mathcal{A}_t \bigcup_{\beta<\alpha} \bigcup_{j \in Y_\beta} a_{t^\frown j}} b_0 \cdots b_{n-1},$$

we have $a \downarrow_{\mathcal{A}_t Z^* b_0 \cdots b_{n-1}} b_n$. Thus by transitivity, $a \downarrow_{\mathcal{A}_t Z^*} b_0 \cdots b_n$.

$\boxminus$

We conclude that $|Y'_\alpha| = |Y_\alpha|$, and items (2) and (3) follow from the fact that $\{a_t^i \mid i \in Y_\alpha\}$ and $\{b_\alpha^i \mid i \in Y'_\alpha\}$ are basis of W and W', respectively. We conclude that $M_\eta^* \models \Phi_\eta^t(\mathcal{A}_t, \mathcal{A}_{t^-}, a_t)$. Following the same argument, we show that $M_\eta^* \models \Phi_\eta^t(\mathcal{A}_t)$ holds for t the root of T. So $M_\eta^* \models \Phi_\eta$.

Notice that above, the well-orders $<^n$ were used only to express the dimensions in the required infinitary logic. Neither of these dimensions nor the formulas that express them depend on the choice of the well-orders, as long as the order type of the orders is κ. Thus $M_\eta^* \models \Phi_\eta$ is independent of the choice of the well-orders in M_η^*.

To prove the second item, let ξ be such that $M_\xi \cong M_\eta$ and $\Pi : M_\eta \to M_\xi$ be an isomorphism. Let us define the following orders, $<_\eta^n$ for $n < \omega$, on M_η:

$$x <_\eta^n y \ \Leftrightarrow\ \Pi(x) <^n \Pi(y)$$

where $<^n$ is the well-ordering of κ^n of order type κ in M_ξ^*. Let M_η^{**} be the expansion of M_η to L^* with the orders $<_\eta^n$ as the interpretation of the well-orderings. Thus Π is an isomorphism between M_ξ^* and M_η^{**}. Since Φ_η is independent of the choice of the well-orders (as far as the order type of the orders is κ), we conclude that $M_\eta^{**} \models \Phi_\eta$. So $M_\xi^* \models \Phi_\eta$, as wanted. $\boxminus$

Corollary 5.15. *Let $\mathcal{T}$ be a countable ω-stable NDOP shallow L-theory. Suppose κ is a regular cardinal such that $\kappa^\omega = \kappa$. If the number of isomorphism classes is at most κ, then $\cong_{\mathcal{T}}$ is κ-Borel.*

Notice that an ω-stable NDOP shallow L-theory is a classifiable shallow theory. By Shelah's Main Gap Theorem, the number of isomorphism classes of a classifiable shallow theory is less than $\beth_{\omega_1}(|\alpha|)$, where $\kappa = \aleph_\alpha$. Thus if $\kappa = \aleph_\alpha$ is a regular cardinal such that $\kappa^\omega = \kappa$ and $\beth_{\omega_1}(|\alpha|) \leq \kappa$, and $\mathcal{T}$ is an ω-stable NDOP shallow L-theory, then $\cong_{\mathcal{T}}$ is κ-Borel.

Remark 5.16. Let $\mathcal{T}$ be a countable stable theory and $\kappa^\omega = \kappa$. Then the set of all η such that $M_\eta \models \mathcal{T}$ is strongly ω-saturated is κ-Borel.
Proof. It is enough to find a $L_{\kappa^+\omega_1}$-sentence φ such that for all η, $M_\eta \models \mathcal{T}$ is strongly ω-saturated if and only if $M_\eta \models \varphi$.

We construct φ as follows. Let $\langle P_i^n \mid i < 2^\omega \rangle$ be the list of all complete n-types $p(x)$ over the empty set. For all $i < 2^\omega$ and $0 < m < \omega$, let $\langle \theta_j^{im} \mid j < \omega \rangle$ be the list of all formulas $\theta(u, v, x)$ such that if $a \models P_i^n$, then $\theta(u, v, a)$ defines a finite equivalence relation on M^m. For all $i < 2^\omega$ and $0 < m < \omega$, let $\langle q_k^{im} \mid k < 2^\omega \rangle$ the list of all complete types $q(x, u_j)_{j<\omega}$ over the empty set such that $q_k^{im} \vdash P_i^n$ and if $(a, a_j)_{j<\omega} \models q_k^{im}$, then there is b such that $\models \theta_j^{im}(a_j, b, a)$ for all $j < \omega$. For all $i < 2^\omega$ and $0 < m < \omega$, let

$$\Psi_m^i = \forall u_0 \forall u_1 \cdots (\bigwedge_{k<2^\omega} (\bigwedge q_k^{im} \to \exists v \bigwedge_{j<\omega} \theta_j^{im}(u_j, v, x)))).$$

Now, for all $n < \omega$ and $0 < m < \omega$, let $\varphi_m^n = \forall x \bigwedge_{i<2^\omega} (\bigwedge P_i^n \to \Psi_m^i)$. Finally, let

$$\varphi = \bigwedge \mathcal{T} \ \wedge \bigwedge_{n<\omega,\ 0<m<\omega} \varphi_m^n.$$

Since for every $M \models \mathcal{T}$ and for every $a \in M^n$, every equivalence class of every equivalence relation definable over a has a representative in M; $M \models \varphi$ implies that M is strongly ω-saturated. The other direction is clear. $\boxminus$

Theorem 5.17. *Let $\mathcal{T}$ be a countable superstable NDOP shallow L-theory. Suppose κ is a regular cardinal such that $\kappa^{(2^\omega)} = \kappa$. If $M_\xi \models \mathcal{T}$ is strongly ω-saturated, then* $\mathrm{Orb}(M_\xi)$ *is κ-Borel.*

The proof of this theorem is essentially the same as that of Theorem 5.4. Only obvious changes are needed, e.g. Morley rank is replaced by U-rank.

6. Open questions

(1) Suppose $2^\omega = 2^{\omega_1} = \omega_2$. What can be said about ω_1-Borelness of (orbits of) models of cardinality $\aleph_1$?

(2) Suppose $\kappa^\omega = \kappa = \mathrm{cf}(\kappa)$. Does it follow that a countable first order theory T is classifiable and shallow if and only if $\cong_T$ is κ-Borel on κ^κ?

(3) Are the following inclusions proper:
   - κ-Borel $\subseteq \kappa$-Δ_1^1.
   - κ-$\Delta_1^1 \subseteq (\kappa, \kappa^{<\kappa})$-Borel*.

(4) Does $(\kappa, \kappa^{<\kappa})$-Borel*= κ-Σ_1^1 imply $\kappa^{<\kappa} = \kappa$.

(5) Do Generalized Baire Spaces have a separation property, under the assumption $\kappa^{<\kappa} > \kappa$?

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