INCLUSION MODULO NONSTATIONARY

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Abstract. A classical theorem of Hechler asserts that the structure $(\omega^\omega, \leq^*)$ is universal in the sense that for any σ-directed poset $\mathbb{P}$ with no maximal element, there is a *ccc* forcing extension in which $(\omega^\omega, \leq^*)$ contains a cofinal order-isomorphic copy of $\mathbb{P}$. In this paper, we prove a consistency result concerning the universality of the higher analogue $\left(\kappa^\kappa, \leq^S\right)$.

Theorem. Assume GCH. For every regular uncountable cardinal κ, there is a cofinality-preserving GCH-preserving forcing extension in which for every analytic quasi-order $\mathbb{Q}$ over κ^κ and every stationary subset S of κ, there is a Lipschitz map reducing $\mathbb{Q}$ to $(\kappa^\kappa, \leq^S)$.

1. Introduction

Recall that a *quasi-order* is a binary relation which is reflexive and transitive. A well-studied quasi-order over the Baire space $\mathbb{N}^\mathbb{N}$ is the binary relation $\leq^*$ which is defined by letting, for any two elements $\eta : \mathbb{N} \to \mathbb{N}$ and $\xi : \mathbb{N} \to \mathbb{N}$,

$$\eta \leq^* \xi \text{ iff } \{n \in \mathbb{N} \mid \eta(n) > \xi(n)\} \text{ is finite.}$$

This quasi-order is behind the definitions of cardinal invariants $\mathfrak{b}$ and $\mathfrak{d}$ (see [Bla10, §2]), and serves as a key to the analysis of *oscillation of real numbers* which is known to have prolific applications to topology, graph theory, and forcing axioms (see [Tod89]). By a classical theorem of Hechler [Hec74], the structure $(\mathbb{N}^\mathbb{N}, \leq^*)$ is universal in that sense that for any σ-directed poset $\mathbb{P}$ with no maximal element, there is a *ccc* forcing extension in which $(\mathbb{N}^\mathbb{N}, \leq^*)$ contains a cofinal order-isomorphic copy of $\mathbb{P}$.

In this paper, we consider (a refinement of) the higher analogue of the relation $\leq^*$ to the realm of the generalized Baire space κ^κ (sometimes refered as the higher Baire space), where κ is a regular uncountable cardinal. This is done by simply replacing the ideal of finite sets with the ideal of nonstationary sets, as follows.[1]

Definition 1.1. Given a stationary subset $S \subseteq \kappa$, we define a quasi-order $\leq^S$ over κ^κ by letting, for any two elements $\eta : \kappa \to \kappa$ and $\xi : \kappa \to \kappa$,

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is nonstationary.}$$

Note that since the nonstationary ideal over S is σ-closed, the quasi-order $\leq^S$ is well-founded, meaning that we can assign a *rank* value $\|\eta\|$ to each element η of κ^κ. The utility of this approach is demonstrated in the celebrated work of Galvin and Hajnal [GH75] concerning the behavior of the power function over the singular cardinals, and, of course, plays an important role in Shelah's *pcf theory* (see [AM10, §4]). It was also demonstrated to be useful in the study of partition relations of singular cardinals of uncountable cofinality [She09].

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[1] A comparison of the generalization considered here with the one obtained by replacing the ideal of finite sets with the ideal of bounded sets may be found in [CS95, §8].

In this paper, we first address the question of how $\leq^S$ compares with $\leq^{S'}$ for various subsets S and S'. It is proved:

Theorem A. *Assume that κ is a regular uncountable cardinal and* GCH *holds. Then there exists a cofinality-preserving* GCH*-preserving forcing extension in which for all stationary subsets S, S' of κ, there exists a map $f : \kappa^{\leq\kappa} \to 2^{\leq\kappa}$ such that, for all $\eta, \xi \in \kappa^{\leq\kappa}$,*

- $\mathrm{dom}(f(\eta)) = \mathrm{dom}(\eta)$*;*
- *if $\eta \subseteq \xi$, then $f(\eta) \subseteq f(\xi)$;*
- *if $\mathrm{dom}(\eta) = \mathrm{dom}(\xi) = \kappa$, then $\eta \leq^S \xi$ iff $f(\eta) \leq^{S'} f(\xi)$.*

Note that as $\mathrm{rng}(f \restriction \kappa^\kappa) \subseteq 2^\kappa$, the above assertion is non-trivial even in the case $S = S' = \kappa$, and forms a contribution to the study of lossless encoding of substructures of $(\kappa^{\leq\kappa}, \ldots)$ as substructures of $(2^{\leq\kappa}, \ldots)$ (see, e.g., [BR17, §7]).

To formulate our next result — an optimal strengthening of Theorem A — let us recall a few basic notions from generalized descriptive set theory. *The generalized Baire space* is the set κ^κ endowed with the bounded topology, in which a basic open set takes the form $[\zeta] := \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$, with ζ, an element of $\kappa^{<\kappa}$. A subset $F \subseteq \kappa^\kappa$ is *closed* iff its complement is open iff there exists a tree $T \subseteq \kappa^{<\kappa}$ such that $[T] := \{\eta \in \kappa^\kappa \mid \forall \alpha < \kappa(\eta \restriction \alpha \in T)\}$ is equal to F. A subset $A \subseteq \kappa^\kappa$ is *analytic* iff there is a closed subset F of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $\mathrm{pr}(F) := \{\eta \in \kappa^\kappa \mid \exists \xi \in \kappa^\kappa \ (\eta, \xi) \in F\}$ is equal to A. *The generalized Cantor space* is the subspace 2^κ of κ^κ endowed with the induced topology. The notions of open, closed and analytic subsets of 2^κ, $2^\kappa \times 2^\kappa$ and $\kappa^\kappa \times \kappa^\kappa$ are then defined in the obvious way.

Definition 1.2. The restriction of the quasi-order $\leq^S$ to 2^κ is denoted by $\subseteq^S$.

For all $\eta, \xi \in \kappa^\kappa$, denote $\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\})$.

Definition 1.3. Let R_1 and R_2 be binary relations over $X_1, X_2 \in \{2^\kappa, \kappa^\kappa\}$, respectively. A function $f : X_1 \to X_2$ is said to be:

(a) a *reduction of R_1 to R_2* iff, for all $\eta, \xi \in X_1$,
$$\eta \ R_1 \ \xi \text{ iff } f(\eta) \ R_2 \ f(\xi).$$
(b) Λ-*Lipschitz* iff $\Lambda \in \kappa$ and, for all $\eta, \xi \in X_1$,
$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)) \cdot \Lambda.$$

The existence of a function f satisfying (a) and (b) is denoted by $R_1 \hookrightarrow_\Lambda R_2$.

In the above language, Theorem A provides a model in which, for all stationary subsets S, S' of κ, $\leq^S \hookrightarrow_1 \subseteq^{S'}$. As $\leq^S$ is an analytic quasi-order over κ^κ, it is natural to ask whether a stronger universality result is possible, and it is moreover forceable that *any* analytic quasi-order over κ^κ admits a 1-Lipschitz reduction to $\subseteq^{S'}$ for some (or maybe even for all) stationary $S' \subseteq \kappa$. The answer turns out to be affirmative, hence the choice of the title of this paper.

Theorem B. *Assume that κ is a regular uncountable cardinal and* GCH *holds. Then there exists a cofinality-preserving* GCH*-preserving forcing extension in which, for every analytic quasi-order Q over κ^κ and every stationary $S \subseteq \kappa$, $Q \hookrightarrow_1 \subseteq^S$.*

Remark. The universality statement under consideration is optimal, as $Q \hookrightarrow_1 \subseteq^S$ implies that Q analytic.

The proof of the preceding goes through a new diamond-type principle for reflecting second-order formulas, introduced here and denoted by $\mathrm{Dl}^*_S(\Pi^1_2)$. This principle is a strengthening of Jensen's $\diamondsuit_S$ and a weakening of Devlin's $\diamondsuit^\sharp_\kappa$. For κ a successor cardinal, we have $\mathrm{Dl}^*_S(\Pi^1_2) \Rightarrow \diamondsuit^*_S$ but not $\diamondsuit^*_S \Rightarrow \mathrm{Dl}^*_S(\Pi^1_2)$ (see Remark 4.2 below). Another crucial difference between the two is that, unlike $\diamondsuit^*_S$, the principle $\mathrm{Dl}^*_S(\Pi^1_2)$ is compatible with the set S being ineffable.

In Section 2, we establish the consistency of the new principle, in fact, proving that it follows from an abstract condensation principle that was introduced and studied in [FH11, HWW15]. It thus follows that it is possible to force $\mathrm{Dl}^*_S(\Pi^1_2)$ to hold over all stationary subsets S of a prescribed regular uncountable cardinal κ. It also follows that, in canonical models for Set Theory (including any $L[E]$ model with Jensen's λ-indexing which is sufficiently iterable and has no subcompact cardinals), $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for every stationary subset S of every regular uncountable (including ineffable) cardinal κ.

Then, in Section 3, the core combinatorial component of our result is proved:

Theorem C. *Suppose S is a stationary subset of a regular uncountable cardinal κ. If $\mathrm{Dl}^*_S(\Pi^1_2)$ holds, then, for every analytic quasi-order Q over κ^κ, $Q \hookrightarrow_1 \subseteq^S$.*

2. A Diamond reflecting second-order formulas

In [Dev82], Devlin introduced a strong form of the Jensen-Kunen principle $\diamondsuit^+_\kappa$, which he denoted by $\diamondsuit^\sharp_\kappa$, and proved:

Fact 2.1 (Devlin, [Dev82, Theorem 5])**.** *In L, for every regular uncountable cardinal κ that is not ineffable, $\diamondsuit^\sharp_\kappa$ holds.*

Remark 2.2. A subset S of a regular uncountable cardinal κ is said to be *ineffable* iff, for every sequence $\langle Z_\alpha \mid \alpha \in S\rangle$, there exists a subset $Z \subseteq \kappa$, for which $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha \cap \alpha\}$ is stationary. Note that the collection of non-ineffable subsets of κ forms a normal ideal that contains $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \alpha\}$ as an element. Also note that if κ is ineffable, then κ is strongly inaccessible.

As said before, in this paper, we consider a refinement of Devlin's principle compatible with κ being ineffable. Devlin's principle as well as its refinement provide us with Π^1_2-reflection over structures of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$. We now describe the relevant logic in detail.

A Π^1_2-sentence ϕ is a formula of the form $\forall X \exists Y \varphi$ where φ is a first-order sentence over a relational language $\mathcal{L}$ as follows:

- $\mathcal{L}$ has a predicate symbol ϵ of arity 2;
- $\mathcal{L}$ has a predicate symbol $\mathbb{X}$ of arity $m(\mathbb{X})$;
- $\mathcal{L}$ has a predicate symbol $\mathbb{Y}$ of arity $m(\mathbb{Y})$;
- $\mathcal{L}$ has infinitely many predicate symbols $(\mathbb{A}_n)_{n\in\omega}$, each $\mathbb{A}_n$ is of arity $m(\mathbb{A}_n)$.

Definition 2.3. For sets N and x, we say that N *sees* x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$.

Suppose that a set N sees an ordinal α, and that $\phi = \forall X \exists Y \varphi$ is a Π^1_2-sentence, where φ is a first-order sentence in the above-mentioned language $\mathcal{L}$. For every sequence $(A_n)_{n\in\omega}$ such that, for all $n \in \omega$, $A_n \subseteq \alpha^{m(\mathbb{A}_n)}$, we write

$$\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models_N \phi$$

to express that the two hold:

1. $(A_n)_{n\in\omega} \in N$;
2. $\langle N, \in\rangle \models (\forall X \subseteq \alpha^{m(\mathbb{X})})(\exists Y \subseteq \alpha^{m(\mathbb{Y})})[\langle \alpha, \in, X, Y, (A_n)_{n\in\omega}\rangle \models \varphi]$, where:
   - $\in$ is the interpretation of ϵ;
   - X is the interpretation of $\mathbb{X}$;
   - Y is the interpretation of $\mathbb{Y}$, and
   - for all $n \in \omega$, A_n is the interpretation of $\mathbb{A}_n$.

Convention 2.4. We write α^+ for $|\alpha|^+$, and write $\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models \phi$ for

$$\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models_{H_{\alpha^+}} \phi.$$

Definition 2.5 (Devlin, [Dev82])**.** Let κ be a regular and uncountable cardinal. $\diamondsuit^\sharp_\kappa$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha < \kappa\rangle$ satisfying the following:

(1) for every infinite $\alpha < \kappa$, N_α is a set of cardinality $|\alpha|$ that sees α;
(2) for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C$, $C \cap \alpha, X \cap \alpha \in N_\alpha$;
(3) whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha < \kappa$ such that $\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.

Consider the following refinement:

Definition 2.6. Let κ be a regular and uncountable cardinal, and $S \subseteq \kappa$ stationary. $\mathrm{Dl}^*_S(\Pi^1_2)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ satisfying the following:

(1) for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;
(2) for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;
(3) whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.

Remark 2.7. The choice of notation for the above principle is motivated by [HLS93, Definition 1.8] and [TV99, Definition 45].

The goal of this section is to derive $\mathrm{Dl}^*_S(\Pi^1_2)$ from an abstract principle which is both forceable and a consequence of $V = L[E]$, for $L[E]$ an iterable extender model with Jensen λ-indexing without a subcompact cardinal (see [SZ01, SZ04]). Note that this covers all $L[E]$ models that can be built so far.

Convention 2.8. The class of ordinals is denoted by OR. The class of ordinals of cofinality μ is denoted by $\mathrm{cof}(\mu)$, and the class of ordinals of cofinality greater than μ is denoted by $\mathrm{cof}(>\mu)$. For a set of ordinals a, we write $\mathrm{acc}(a) := \{\alpha \in a \mid \sup(a \cap \alpha) = \alpha > 0\}$. ZF^- denotes ZF without the power-set axiom, and $r(\alpha)$ denotes a formula expressing that "α is regular". The transitive closure of a set X is denoted by $\mathrm{trcl}(X)$, and the Mostowski collapse of a structure $\mathfrak{B}$ is denoted by $\mathrm{clps}(\mathfrak{B})$.

Convention 2.9. Whenever λ is a limit ordinal, and $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ is a $\subseteq$-increasing, continuous sequence of sets, we denote its limit $\bigcup_{\beta<\lambda} M_\beta$ by M_λ.

Definition 2.10 (Friedman-Holy [FH11], Holy-Welch-Wu [HWW15])**.** Let λ be a cardinal of uncountable cofinality or the class OR of all ordinals. We say that $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ is a witness to the fact that *local club condensation holds in* (η, ζ), and denote this by $\langle H_\lambda, \in, \vec{M}\rangle \models \mathsf{LCC}(\eta, \zeta)$, iff all of the following hold true:

(1) $\eta < \zeta \leq \lambda + 1$;
(2) $\vec{M}$ is nice filtration of H_λ:
  (a) for all $\beta < \lambda$, M_β is a transitive set with $M_\beta \cap \mathrm{OR} = \beta$;
  (b) $\vec{M}$ is $\in$-increasing, that is, $\alpha < \beta < \lambda \implies M_\alpha \in M_\beta$;
  (c) $\vec{M}$ is continuous, that is, for every limit ordinal $\beta < \lambda$, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$;
  (d) $M_\lambda = H_\lambda$.[2]
(3) For every ordinal α in the interval (η, ζ) and every sequence $\mathcal{F} = \langle (F_n, k_n) \mid n \in \omega\rangle$ such that, for all $n \in \omega$, $k_n \in \omega$ and $F_n \subseteq (M_\alpha)^{k_n}$, there is a sequence $\vec{\mathfrak{B}} = \langle \mathcal{B}_\beta \mid \beta < |\alpha|\rangle$ having the following properties:

[2]Recall Convention 2.9.

(a) for all $\beta < |\alpha|$, $\mathcal{B}_\beta$ is of the form $\langle B_\beta, \in, \vec{M}\restriction(B_\beta \cap \mathrm{OR}), (F_n \cap (B_\beta)^{k_n})_{n\in\omega}\rangle$;
(b) for all $\beta < |\alpha|$, $\mathcal{B}_\beta \prec \langle M_\alpha, \in, \vec{M}\restriction\alpha, (F_n)_{n\in\omega}\rangle$;[3]
(c) for all $\beta < |\alpha|$, $\beta \subseteq B_\beta$ and $|B_\beta| = |\beta|$;
(d) for all $\beta < |\alpha|$, there exists $\bar{\beta} < \lambda$ such that

$$\mathrm{clps}(\langle B_\beta, \in, \langle B_\delta \mid \delta \in B_\beta \cap \mathrm{OR}\rangle\rangle) = \langle M_{\bar{\beta}}, \in, \langle M_\delta \mid \delta \in \bar{\beta}\rangle\rangle;$$

(e) $\langle B_\beta \mid \beta < |\alpha|\rangle$ is $\subseteq$-increasing, continuous and converging to M_α.

For $\vec{\mathfrak{B}}$ as in Clause (3) above we say that $\vec{\mathfrak{B}}$ *witnesses* $\mathsf{LCC}(\eta,\zeta)$ *at* α *with respect to* $\mathcal{F}$. We write $\mathsf{LCC}(\eta,\zeta]$ for $\mathsf{LCC}(\eta,\zeta+1)$ and $\mathsf{LCC}(\eta)$ for $\mathsf{LCC}(\eta,\lambda)$.

Remark 2.11. There are first-order sentences $\psi_0(\dot{\eta})$ and $\psi_1(\dot{\eta},\dot{\zeta})$ in the language $\mathcal{L} := \{\in, \vec{M}, \dot{\eta}, \dot{\zeta}\}$ of set theory augmented by a predicate for a nice filtration and two ordinals such that, for $\eta < \zeta \leq \lambda+1$, if we interpret $\dot{\eta} = \eta$ and $\dot{\zeta} = \zeta$, then

- $(\langle H_\lambda, \in, \vec{M}\rangle \models \psi_0(\eta)) \Leftrightarrow (\langle H_\lambda, \in, \vec{M}\rangle \models \mathsf{LCC}(\eta))$, and
- $(\langle H_\lambda, \in, \vec{M}\rangle \models \psi_1(\eta,\zeta)) \Leftrightarrow (\langle H_\lambda, \in, \vec{M}\rangle \models \mathsf{LCC}(\eta,\zeta))$.

Fact 2.12 (Holy-Welch-Wu, [HWW15, pp. 1362 and §4])**.** *Assume* GCH*. For every regular cardinal* κ*, there is a (set-size) notion of forcing* $\mathbb{P}$ *which is* $({<}\kappa)$*-directed-closed and has the* κ^+*−cc such that, in* $V^{\mathbb{P}}$*, the two holds:*

(1) there is $\vec{M}$ *such that* $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa,\kappa^+]$*, and*
(2) there is a Δ_1*-formula* Θ *and a parameter* $a \subseteq \kappa$ *such that the order defined by* $x <_\Theta y \leftrightarrow H_{\kappa^+} \models \Theta(x,y,a)$ *is a global well-order of* H_{κ^+}*.*

By reading [SZ04, Theorem 0.1] and the proof of [FH11, Theorem 8], one arrives at the following conclusion.

Lemma 2.13. *Suppose* $L[E]$ *is an iterable extender model with Jensen* λ*-indexing. Then the following are equivalent:*

(1) $\langle L[E], \in, \langle L_\beta[E] \mid \beta \in \mathrm{OR}\rangle\rangle \models \mathsf{LCC}(\aleph_0)$*;*
(2) $\langle L[E], \in\rangle \models$ *there exist no subcompact cardinals.* □

Lemma 2.14. *Suppose* $\vec{M}$ *is such that* $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa,\kappa^+]$*. Then:*

(1) for every cardinal $\mu < \kappa^+$*,* $H_\mu = M_\mu$*;*
(2) for every ordinal $\delta \leq \kappa^+$*,* $|M_\delta| = |\delta|$*;*
(3) there are club many $\delta < \kappa^+$ *such that* $\langle M_\delta, \in, \vec{M}\restriction\delta\rangle \prec \langle M_{\kappa^+}, \in, \vec{M}\rangle$*.*

Proof. This follows from the arguments of [HWW15, Theorem 3.1]. For the reader's convenience, we include a proof of Clauses (1) and (3).

(1) It suffices to prove it for μ successor, say $\mu = \theta^+$.

- ▶ $M_\mu \subseteq H_\mu$: Let $\vec{\mathfrak{B}}$ witness $\mathsf{LCC}(\kappa,\kappa^+]$ at κ^+ with respect to $\mathcal{F} := \emptyset$. For each $\alpha < \mu$, let $\beta(\alpha) < \kappa^+$ be such that $\mathrm{clps}(\mathfrak{B}_\alpha) = \langle M_{\beta(\alpha)}, \in, \ldots\rangle$. By Clauses (2)(a) and (3)(c) of Definition 2.10, we have $M_{\beta(\alpha)} \cap \mathrm{OR} = \beta(\alpha)$ and $|M_{\beta(\alpha)}| = |B_\alpha| = |\alpha| < \mu$, so that $\beta(\alpha) < \mu$. It thus follows that $Y := \{\beta(\alpha) \mid \alpha < \mu\}$ is cofinal in μ and, as each M_β is transitive,

$$M_\mu = \bigcup_{\beta<\mu} M_\beta = \bigcup_{\beta\in Y} M_\beta \subseteq H_\mu.$$

- ▶ $H_\mu \subseteq M_\mu$: Let $x \in H_\mu$ be arbitrary. Fix a surjection $f : \theta \to \mathrm{trcl}(\{x\})$. Let $\vec{\mathfrak{B}}$ witness $\mathsf{LCC}(\kappa,\kappa^+]$ at κ^+ with respect to $\mathcal{F} := \langle(f,2)\rangle$. For notational simplicity, we write $\mathcal{F}_0$ for f. Let $\beta < \kappa^+$ be such that

[3] Note that the case $\alpha = \lambda$ uses Convention 2.9.

clps($\mathfrak{B}_{\theta+1}$) $= \langle M_\beta, \in, \dots\rangle$. By Definition 2.10(3)(c), $\theta+1 \subseteq B_{\theta+1}$, so that, altogether, $\theta < \beta < \mu$. Now, as

$$\mathfrak{B}_{\theta+1} \prec \langle H_{\kappa^+}, \in, \vec{M}, \mathcal{F}_0\rangle \models \exists y(\forall\gamma\forall\delta(\mathcal{F}_0(\gamma,\delta) \leftrightarrow (\gamma,\delta) \in y)),$$

we have $f \in B_{\theta+1}$. Since $\mathrm{dom}(f) \subseteq B_{\theta+1}$, $\mathrm{rng}(f) \subseteq B_{\theta+1}$. But $\mathrm{rng}(f) = \mathrm{trcl}(\{x\})$ is a transitive set, so that the Mostowski collapsing map $\pi : B_{\theta+1} \to M_\beta$ is the identity over $\mathrm{trcl}(\{x\})$, meaning that $x \in \mathrm{trcl}(\{x\}) \subseteq M_\beta \subseteq M_\mu$.

(3) Let $\vec{\mathcal{B}}$ witness $\mathsf{LCC}(\kappa,\kappa^+]$ at κ^+ with respect to $\mathcal{F} := \emptyset$. By continuity of the sequences $\langle B_\delta \mid \delta < \kappa^+\rangle$ and $\langle M_\delta \mid \delta < \kappa^+\rangle$, the set $D := \{\delta < \kappa^+ \mid B_\delta = M_\delta\}$ is closed. We shall prove that D is unbounded, and then the conclusion will follow from Clause (3)(b) of Definition 2.10. Let $\varepsilon < \kappa^+$ be arbitrary, and we shall find $\delta \in D$ above ε. As $\bigcup_{\beta<\kappa^+} B_\beta = M_{\kappa^+} = \bigcup_{\beta<\kappa^+} M_\beta$ with $|B_\beta| = |\beta| = |M_\beta|$ for all $\beta < \kappa^+$, and as $|M_{\kappa^+}| = \kappa^+$, we can recursively construct two sequences of ordinals $\langle \gamma_n \mid n < \omega\rangle$ and $\langle \delta_n \mid n < \omega\rangle$ such that, for all $n < \omega$:

- $\varepsilon < \gamma_n < \delta_n < \gamma_{n+1} < \kappa^+$, and
- $M_{\gamma_n} \subseteq B_{\delta_n} \subseteq M_{\gamma_{n+1}}$,

so that the two sequences of ordinals converge to the same ordinal, say δ, and, by continuity,

$$M_\delta = \bigcup_{n<\omega} M_{\gamma_n} = \bigcup_{n<\omega} B_{\delta_n} = B_\delta.$$

Altogether, $\delta \in D \setminus (\varepsilon+1)$. □

Theorem 2.15. *Suppose that κ is a regular uncountable cardinal, and $\vec{M}$ is such that $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa,\kappa^+]$. Suppose further that there is a subset $a \subseteq \kappa$ and a formula $\Theta \in \Sigma_\omega$ which defines a well-order $<_\Theta$ in H_{κ^+} via $x <_\Theta y$ iff $H_{\kappa^+} \models \Theta(x,y,a)$. Then, for every stationary $S \subseteq \kappa$, $\mathrm{Dl}^*_S(\Pi^1_2)$ holds.*

Proof. Let $S' \subseteq \kappa$ be stationary. We shall prove that $\mathrm{Dl}^*_{S'}(\Pi^1_2)$ holds by adjusting Devlin's proof of Fact 2.1.

As a first step, we identify a subset S of S' of interest.

Claim 2.15.1. *There exists a stationary non-ineffable subset $S \subseteq S' \setminus \omega$ such that, for every $\alpha \in S' \setminus S$, $|H_{\alpha^+}| < \kappa$.*

Proof. If S' is non-ineffable, then let $S := S' \setminus \omega$, so that $H_{\alpha^+} = H_\omega$ for all $\alpha \in S' \setminus S$. From now on, suppose that S' is ineffable. In particular, κ is strongly inaccessible and $|H_{\alpha^+}| < \kappa$ for every $\alpha < \kappa$. Let $S := S' \setminus (\omega \cup T)$, where

$$T := \{\alpha < \kappa \cap \mathrm{cof}(>\omega) \mid S' \cap \alpha \text{ is stationary in } \alpha\}.$$

To see that S is stationary, let E be an arbitrary club in κ.

► If $S' \cap \mathrm{cof}(\omega)$ is stationary, then since $S' \cap \mathrm{cof}(\omega) \subseteq S$, we infer that $S \cap E \neq \emptyset$.

► If $S' \cap \mathrm{cof}(\omega)$ is non-stationary, then fix a club $C \subseteq E$ disjoint from $S' \cap \mathrm{cof}(\omega)$, and let $\alpha := \min(\mathrm{acc}(C) \cap S')$. Then $\mathrm{cf}(\alpha) > \omega$ and $C \cap \alpha$ is a club in α disjoint from S', so that $\alpha \notin T$. Altogether, $\alpha \in S \cap E$.

To see that S is non-ineffable, we define a sequence $\langle Z_\alpha \mid \alpha \in S\rangle$, as follows. For every $\alpha \in S$, fix a closed and cofinal subset Z_α of α with $\mathrm{otp}(Z_\alpha) = \mathrm{cf}(\alpha)$ such that, if $\mathrm{cf}(\alpha) > \omega$, then the club Z_α is disjoint from $S' \cap \alpha$. Towards a contradiction, suppose that $Z \subseteq \kappa$ is a set for which $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha\}$ is stationary. Clearly, Z is closed and cofinal in κ, so that $Z \cap S'$ is stationary, $\mathrm{otp}(Z \cap S') = \kappa$ and hence $E := \{\alpha < \kappa \mid \mathrm{otp}(Z \cap S' \cap \alpha) = \alpha > \omega\}$ is a club. Pick $\alpha \in E \cap S$ such that $Z \cap \alpha = Z_\alpha$. As

$$\mathrm{cf}(\alpha) = \mathrm{otp}(Z_\alpha) = \mathrm{otp}(Z \cap \alpha) \geq \mathrm{otp}(Z \cap S' \cap \alpha) = \alpha > \omega,$$

it must be the case that Z_α is a club disjoint from $S' \cap \alpha$, while $Z_\alpha = Z \cap \alpha$ and $Z \cap S' \cap \alpha \neq \emptyset$. This is a contradiction. $\square$

Let S be given by the preceding claim. We shall focus on constructing a sequence $\langle N_\alpha \mid \alpha \in S\rangle$ witnessing $\mathrm{Dl}^*_S(\Pi^1_2)$ such that, in addition, $|N_\alpha| = |\alpha|$ for every $\alpha \in S$. It will then immediately follow that the sequence $\langle N'_\alpha \mid \alpha \in S'\rangle$ defined by letting $N'_\alpha := N_\alpha$ for $\alpha \in S$, and $N'_\alpha := H_{\alpha^+}$ for $\alpha \in S' \setminus S$ will witness the validity of $\mathrm{Dl}^*_{S'}(\Pi^1_2)$.

Here we go. As S is non-ineffable, fix a sequence $\vec{Z} = \langle Z_\alpha \mid \alpha \in S\rangle$ with $Z_\alpha \subseteq \alpha$ for all $\alpha \in S$, such that, for every $Z \subseteq \kappa$, $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha\}$ is nonstationary.

As we have a sequence $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ such that $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa, \kappa^+]$, for each $\alpha \in S$, we may define S_α to be the set of all $\beta \in \alpha^+$ satisfying the following list of conditions:

i) $\langle M_\beta, \in, \vec{M} \restriction \beta\rangle \models \mathsf{LCC}(\alpha)$,
ii) $\langle M_\beta, \in\rangle \models \mathsf{ZF}^-$ & α is the largest cardinal,
iii) $\langle M_\beta, \in\rangle \models r(\alpha)$ & $S \cap \alpha$ is stationary,
iv) $\langle M_\beta, \in\rangle \models \Theta(x, y, a \cap \alpha)$ defines a global well-order,
v) $\vec{Z} \restriction (\alpha + 1) \notin M_\beta$.

Then, consider the set

$$D := \{\alpha \in S \mid S_\alpha \neq \emptyset \text{ \& } S_\alpha \text{ has no largest element}\}.$$

Define a function $f : S \to \kappa$ as follow. For every $\alpha \in D$, let $f(\alpha) := \sup(S_\alpha)$; for every $\alpha \in S \setminus D$, let $f(\alpha)$ be the least $\gamma < \kappa$ such that M_γ sees α, and $\vec{Z} \restriction (\alpha+1) \in M_\gamma$.

Claim 2.15.2. *f is well-defined. Furthermore, for all $\alpha \in S$, $\alpha < f(\alpha) < \alpha^+$.*

Proof. Let $\alpha \in S$ be arbitrary.

► Suppose $\alpha \in D$. By Lemma 2.14(1), $\bigcup_{\beta<\alpha^+} M_\beta = M_{\alpha^+} = H_{\alpha^+}$, thus there exists $\beta < \alpha^+$ such that $\vec{Z} \restriction (\alpha + 1) \in M_\beta$ and hence condition (v) in the definition of S_α implies that $f(\alpha) \leq \beta < \alpha^+$.

► Suppose $\alpha \notin D$. We need to find some $\gamma < \alpha^+$ such that M_γ sees α, and $\vec{Z} \restriction (\alpha + 1) \in M_\gamma$. Let $\vec{\mathfrak{B}}$ witness $\mathsf{LCC}(\kappa, \kappa^+]$ at κ^+ with respect to $\mathcal{F} := \emptyset$. As in the previous case, we can find an infinite $\beta < \alpha^+$ such that $\vec{Z} \restriction (\alpha + 1) \in M_\beta$. Now, let $\gamma < \kappa^+$ be such that $\mathrm{clps}(\mathfrak{B}_{\beta+1}) = \langle M_\gamma, \in, \ldots\rangle$. By Clauses (2)(a) and (3)(c) of Definition 2.10, $M_\gamma \cap \mathrm{OR} = \gamma$ and $|M_\gamma| = |B_{\beta+1}| = |\beta + 1| < \alpha^+$, so that $\gamma < \alpha^+$. Also, by Clause (3)(c) of Definition 2.10, $\beta + 1 \subseteq B_{\beta+1}$, so that $\beta + 1 \subseteq M_\gamma$ and $\vec{Z} \restriction (\alpha + 1) \in M_\beta \subseteq M_\gamma$. Finally, as $\langle B_{\beta+1}, \in\rangle \prec \langle M_{\kappa^+}, \in\rangle$ and the latter is a model of ZF^-, the Mostowski collapse of the former is p.r.-closed. Recalling that $\alpha + 1 < \beta < \gamma$, we altogether infer that M_γ sees α. $\square$

Define $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ by letting $N_\alpha := M_{f(\alpha)}$ for all $\alpha \in S$. It follows from the preceding Claim together with Lemma 2.14(2) that $|N_\alpha| = |\alpha|$ for all $\alpha \in S$.

In the course of the rest of the proof, we shall occasionally take witnesses to $\mathsf{LCC}(\kappa, \kappa^+]$ with respect to a finite sequence $\mathcal{F} = \langle (F_n, k_n) \mid n \in 4\rangle$; for this, we introduce the following piece of notation:

$$\mathcal{F}_X := \langle (X, 1), (a, 1), (S, 1), (\vec{Z}, 2)\rangle.$$

Claim 2.15.3. *Let $X \subseteq \kappa$. Then there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$.*

Proof. Let $\vec{\mathcal{B}} = \langle \mathcal{B}_\alpha \mid \alpha < \kappa^+\rangle$ witness $\mathsf{LCC}(\kappa, \kappa^+]$ at κ^+ with respect to $\mathcal{F}_X$.

For each $\alpha < \kappa$, let $\beta(\alpha)$ be such that $\mathrm{clps}(\mathfrak{B}_\alpha) = \langle M_{\beta(\alpha)}, \in, \ldots\rangle$, and let $j_\alpha : M_{\beta(\alpha)} \to B_\alpha$ denote the inverse of the collapsing map. Let

$$C := \{\alpha < \kappa \mid B_\alpha \cap \kappa = \alpha\}.$$

Subclaim 2.15.3.1. *C is a club.*

Proof. To see that C is closed in κ, fix an arbitrary $\alpha<\kappa$ with $\sup(C\cap\alpha)=\alpha>0$. As $\langle B_\beta\mid\beta<\kappa^+\rangle$ is $\subseteq$-increasing and continuous, we have

$$\alpha=\bigcup_{\beta\in(C\cap\alpha)}\beta=\bigcup_{\beta\in(C\cap\alpha)}(B_\beta\cap\kappa)=\bigcup_{\beta<\alpha}(B_\beta\cap\kappa)=B_\alpha\cap\kappa.$$

To see that C is unbounded in κ, fix an arbitrary $\varepsilon<\kappa$, and we shall find $\alpha\in C$ above ε. Recall that, by Clause (3)(c) of Definition 2.10, for each $\beta<\kappa$, $\beta\subseteq B_\beta$ and $|B_\beta|=|\beta|<\kappa$. It follow that we may recursively construct an increasing sequence of ordinals $\langle\alpha_n\mid n<\omega\rangle$ such that:

- $\alpha_0:=\sup(B_\varepsilon\cap\kappa)$, and, for all $n<\omega$:
- $\sup(B_{\alpha_n}\cap\kappa)<\alpha_{n+1}<\kappa$.

In particular, $\sup(B_{\alpha_n}\cap\kappa)\in\alpha_{n+1}$ for all $n<\omega$. Consequently, for $\alpha:=\sup_{n<\omega}\alpha_n$, we have that $\alpha<\kappa$, and

$$B_\alpha\cap\kappa=\bigcup_{n<\omega}(B_{\alpha_n}\cap\kappa)\leq\bigcup_{n<\omega}\alpha_{n+1}\leq\bigcup_{n<\omega}(B_{\alpha_{n+2}}\cap\kappa)=\alpha,$$

so that $\alpha\in C\setminus(\varepsilon+1)$. ☐

To see that the club C is as sought, let $\alpha\in C\cap S$ be arbitrary, and we shall verify that $X\cap\alpha\in N_\alpha$.

Since $\vec{\mathcal{B}}$ witnesses $\mathsf{LCC}(\kappa,\kappa^+]$ at κ^+ with respect to $\mathcal{F}_X$, for each Y in $\{X,a,S\}$, we have that

$$\langle B_\alpha,\in,Y\cap B_\alpha\rangle\prec\langle M_{\kappa^+},\in,Y\rangle\models\exists y((z\in y)\leftrightarrow(z\in\kappa\wedge Y(z))),$$

therefore each of X,a,S is a definable element of B_α. So, as, for all $Y\in B_\alpha\cap\mathcal{P}(\kappa)$, $j_\alpha^{-1}(Y)=Y\cap\alpha$, we infer that $X\cap\alpha$, $a\cap\alpha$, and $S\cap\alpha$ are all in $M_{\beta(\alpha)}$. We will show that $\beta(\alpha)<f(\alpha)$, from which it will follow that $X\cap\alpha\in N_\alpha$.

Subclaim 2.15.3.2. *$\beta(\alpha)<f(\alpha)$*

Proof. The analysis splits into two cases: $\alpha\in D$ and $\alpha\notin D$.

► Suppose $\alpha\in D$. As $\mathfrak{B}_\alpha\prec\langle M_{\kappa^+},\in,\vec{M},\mathcal{F}_X\rangle$ and $\mathrm{rng}(j_\alpha)=B_\alpha$, we infer that j_α forms an elementary embedding from $\langle M_{\beta(\alpha)},\in,\ldots\rangle$ to $\langle M_{\kappa^+},\in,\vec{M},\mathcal{F}_X\rangle$ with $j_\alpha(\alpha)=\kappa$. As we have

I) $\langle M_{\kappa^+},\in,\vec{M}\restriction\kappa\rangle\models\mathsf{LCC}(\kappa)$,
II) $\langle M_{\kappa^+},\in\rangle\models\mathsf{ZF}^-$ & κ is the largest cardinal,
III) $\langle M_{\kappa^+},\in\rangle\models r(\kappa)$ & $S\cap\kappa$ is stationary,
IV) $\langle M_{\kappa^+},\in\rangle\models\Theta(x,y,a\cap\kappa)$ defines a global well-order.

it follows that $\beta(\alpha)$ satisfies clauses (i),(ii),(iii) and (iv) of the definition of S_α.

It remains to show that $\vec{Z}\restriction(\alpha+1)\notin M_{\beta(\alpha)}$, and it will follow that $\beta(\alpha)\in S_\alpha$. Towards a contradiction, suppose that $\vec{Z}\restriction(\alpha+1)\in M_{\beta(\alpha)}$. We have

$$\langle M_{\kappa^+},\in\rangle\models\forall Z\subseteq\kappa\ \exists E\text{ club in }\kappa\ (\forall\gamma\in E\cap S\rightarrow Z\cap\gamma\neq Z_\gamma),$$

and hence

$$\langle M_{\beta(\alpha)},\in\rangle\models\forall Z\subseteq\alpha\ \exists E\text{ club in }\alpha\ (\forall\gamma\in E\cap S\rightarrow Z\cap\gamma\neq Z_\gamma).$$

In particular, using $Z:=Z_\alpha$, we find some E such that

$$\langle M_{\beta(\alpha)},\in\rangle\models E\text{ is a club in }\alpha\ (\forall\gamma\in E\cap S\rightarrow Z_\alpha\cap\gamma\neq Z_\gamma).$$

Let $E^*:=j_\alpha(E)$ and $Z^*:=j_\alpha(Z_\alpha)$, so that

$$\langle M_{\kappa^+},\in\rangle\models E^*\text{ is a club in }\kappa\ (\forall\gamma\in E^*\cap S\rightarrow Z^*\cap\gamma\neq Z_\gamma).$$

Then $Z^* \cap \alpha = j_\alpha(Z_\alpha) \cap \alpha = Z_\alpha$, and hence $\alpha \notin E^*$ (recall that $\alpha \in S$). Likewise $E^* \cap \alpha = j_\alpha(E) \cap \alpha = E$, and hence $\alpha \in \mathrm{acc}(E^*) \subseteq E^*$. This is a contradiction.

► If $\alpha \notin D$, then the above argument shows that for every ordinal $\gamma < \kappa$ with $\vec{Z} \restriction (\alpha+1) \in M_\gamma$, we have $\gamma > \beta(\alpha)$, so that $\beta(\alpha) < f(\alpha)$. □

This completes the proof of Claim 2.15.3. □

We are left with addressing Clause (3) of Definition 2.6.

Claim 2.15.4. *The sequence $\langle N_\alpha \mid \alpha \in S\rangle$ reflects Π^1_2-sentences.*

Proof. We need to show that whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with $\phi = \forall X \exists Y \varphi$ a Π^1_2-sentence, for every club $E \subseteq \kappa$, there is $\alpha \in E \cap S$, such that

$$\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi.$$

But by adding E to the list $(A_n)_{n\in\omega}$ of predicates, and by slightly extending the first-order formula φ to also assert that E is unbounded, we would get that any ordinal α satisfying the above equation will also satisfy that α is an accumulation point of the closed set E, so that $\alpha \in E$. It follows that if any Π^1_2-sentence valid in a structure of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$ reflects to some ordinal $\alpha' \in S$, then any Π^1_2-sentence valid in a structure of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$ reflects stationarily often in S.

Thus, let $\vec{A} = (A_n)_{n\in\omega}$, be a sequence of finitary predicates on κ, and let φ be a first-order sentence in the language of $\langle \kappa, \in, \vec{A}, X, Y\rangle$, where $X \subseteq \kappa^p$, $Y \subset \kappa^q$ for some integers p, q, such that $\langle \kappa, \in, \vec{A}\rangle \models \forall X \exists Y \varphi$. Note that by Convention 2.4 and since $M_{\kappa^+} = H_{\kappa^+}$, this means that

$$\langle \kappa, \in, \vec{A}\rangle \models_{M_{\kappa^+}} \forall X \exists Y \varphi.$$

Let γ be the least ordinal such that $\vec{Z}, \vec{A}, S \in M_\gamma$. Note that $\kappa < \gamma < \kappa^+$. Let $\mathcal{L}$ be the first-order language of Set Theory augmented by a predicate $\dot{\vec{M}}$ and constants $\dot{\gamma}, \dot{a}, \dot{\vec{Z}}, \dot{\kappa}, \dot{S}, \dot{A}_n$ for $n \in \omega$, and let T be the theory consisting of the following axioms:

A) $\mathsf{LCC}(\dot{\kappa})$,
B) ZF^- & $\dot{\kappa}$ is the largest cardinal,
C) $r(\dot{\kappa})$ & $\dot{S}$ is stationary in $\dot{\kappa}$,
D) $\Theta(x, y, \dot{a})$ defines a global well-order,
E) $\langle \dot{\kappa}, \in, (\dot{A}_n)_{n\in\omega}\rangle \models \forall X \exists Y \varphi$,
F) $\dot{\vec{Z}}$ witness that $\dot{S}$ is not ineffable,
G) $\dot{\gamma}$ is the least such that $\{\dot{\vec{Z}}, (\dot{A}_n)_{n\in\omega}, \dot{S}\} \in \dot{\vec{M}}(\dot{\gamma})$.

Let Δ denote the set of all $\delta \leq \kappa^+$ such that $\delta > \gamma$ and $\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models T$ where $\dot{\gamma}, \dot{a}, \dot{\vec{Z}}, \dot{\kappa}, \dot{S}, \dot{A}_n$ for $n \in \omega$ are interpreted as $\gamma, a, \vec{Z}, \kappa, S, A_n$ for $n \in \omega$, and $\dot{\vec{M}}$ as $\vec{M} \restriction \delta$. In other words, Δ denotes the set of all $\delta \leq \kappa^+$ such that:

a) $\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models \mathsf{LCC}(\kappa)$,[4]
b) $\langle M_\delta, \in\rangle \models \mathsf{ZF}^-$ & κ is the largest cardinal,
c) $\langle M_\delta, \in\rangle \models r(\kappa)$ & S is stationary in κ,
d) $\langle M_\delta, \in\rangle \models \Theta(x, y, a)$ defines a global well-order,
e) $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models_{M_\delta} \forall X \exists Y \varphi$,
f) $\langle M_\delta, \in\rangle \models \vec{Z}$ witness that S is not ineffable, and
g) $\delta > \gamma$.

[4]In particular, $\delta > \kappa$.

By the fact that $\delta := \kappa^+$ satisfies Clauses (a)–(g) above, it follows from Lemma 2.14(3) that $\mathrm{otp}(\Delta \cap \kappa^+) = \kappa^+$, so we may let $\{\delta_n \mid n < \omega\}$ denote the increasing enumeration of the first ω many elements of Δ.

Let $n < \omega$. As $\langle M_{\delta_{n+1}}, \in\rangle \models |\delta_n| = \kappa$, we may fix in $M_{\delta_{n+1}}$ a sequence $\vec{\mathfrak{B}}_n = \langle \mathcal{B}_{n,\alpha} \mid \alpha < \kappa\rangle$ witnessing $\mathsf{LCC}(\kappa, \kappa^+]$ at δ_n with respect to $\mathcal{F}_\emptyset$ such that, moreover,

$$\langle M_{\delta_{n+1}}, \in, \vec{M} \restriction \delta_{n+1}\rangle \models \text{“}\vec{\mathfrak{B}}_n \text{ is the } <_\Theta\text{-least such witness”}.$$

For every $n < \omega$, let $C_n := \{\alpha < \kappa \mid B_{n,\alpha} \cap \kappa = \alpha\}$. Then, let

$$\alpha' := \min((\bigcap_{n\in\omega} C_n) \cap S).$$

For every $n < \omega$, let β_n be such that $\mathrm{clps}(\mathfrak{B}_{n,\alpha'}) = \langle M_{\beta_n}, \in, \ldots\rangle$.

Since for each formula $\varphi \in T$ and every ordinal $\delta < \kappa^+$, we have that

$$\text{“}\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models \varphi\text{”}$$

is a $\Delta_1^{\mathsf{ZF}^-}$ formula on the parameters $\delta, \vec{M}, \gamma, a, \vec{Z}, \kappa, S, (A_n)_{n\in\omega}, \varphi$,[5] it follows that

$$\text{“}\forall\varphi(\varphi \in T \to \langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models \varphi)\text{”} \tag{1}$$

is a $\Delta_1^{\mathsf{ZF}^-}$ formula in the same parameters plus T. Assuming the formulae were arithmetized in a sufficiently simple way that $T \subseteq V_\omega$, it follows that $T \in H_{\omega_1} = M_{\omega_1}$, so that $T \in M_{\delta_n}$ for every $n < \omega$.

As $M_{\delta_{n+1}}$ is transitive and as the formula of Equation (1) is $\Delta_1^{\mathsf{ZF}^-}$, it follows that, for all $\delta \in M_{\delta_{n+1}} \cap \mathrm{OR}$,

$$(\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models_{\langle M_{\delta_{n+1}}, \in, \vec{M}\restriction\delta_{n+1}\rangle} T), \text{with } \dot{\vec{M}} \text{ interpreted as } \vec{M} \restriction \delta$$

iff

$$(\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models T), \text{with } \dot{\vec{M}} \text{ interpreted as } \vec{M} \restriction \delta.$$

Thus $M_{\delta_{n+1}}$ believes that there are exactly n ordinals δ such that Clauses (a)–(g) hold for M_δ, i.e.

$$\langle M_{\delta_{n+1}}, \in, \vec{M}\restriction\delta_{n+1}\rangle \models \text{“}|\{\delta \mid \langle M_\delta, \in, \vec{M}\restriction\delta\rangle \models T \text{ with } \dot{\vec{M}} \text{ interpreted as } \vec{M}\restriction\delta\}| = n\text{”},$$

while M_{δ_n} believes that there are exactly $n - 1$ such ordinals.

Our next task is to show that the above discussion about $M_{\delta_{n+1}}$ and M_{δ_n} works also for $M_{\beta_{n+1}}$ and M_{β_n}. For this, let $j_n : M_{\beta_n} \to B_{n,\alpha'}$ denote the inverse of the Mostowski collapse.

Subclaim 2.15.4.1. *Let $n \in \omega$. Then $j_n^{-1}(\gamma) = j_0^{-1}(\gamma)$.*

Proof. Since $j_n^{-1}(\vec{Z}) = \vec{Z} \restriction \alpha'$, $j_n^{-1}(\vec{A}) = \vec{A} \restriction \alpha'$ and $j_n^{-1}(S) = S \cap \alpha'$, it follows from

$$\langle M_{\delta_n}, \in, \vec{M} \restriction \delta_n\rangle \models \gamma \text{ is the least ordinal with } \{\vec{Z}, \vec{A}, S\} \subseteq M_\gamma,$$

that

$$\langle M_{\beta_n}, \in, \vec{M} \restriction \beta_n\rangle \models j_n^{-1}(\gamma) \text{ is the least ordinal with } \{\vec{Z} \restriction \alpha', \vec{A} \restriction \alpha', S \cap \alpha'\} \subseteq M_\gamma.$$

Now, let $\bar{\gamma}$ be such that

$$\langle M_{\beta_0}, \in, \vec{M} \restriction \beta_0\rangle \models \bar{\gamma} \text{ is the least ordinal such that } \{\vec{Z} \restriction \alpha', \vec{A} \restriction \alpha', S \cap \alpha'\} \subseteq M_{\bar{\gamma}}.$$

Since $\vec{M}$ is continuous, it follows that $\bar{\gamma}$ is a successor ordinal, that is, $\bar{\gamma} = \sup(\bar{\gamma})+1$. So $\langle M_{\beta_0}, \in, \vec{M} \restriction \beta_0\rangle$ satisfies the conjunction of the two:

- $\{\vec{Z} \restriction \alpha', \vec{A} \restriction \alpha', S \cap \alpha'\} \subseteq M_{\bar{\gamma}}$, and
- $\{\vec{Z} \restriction \alpha', \vec{A} \restriction \alpha', S \cap \alpha'\} \not\subseteq M_{\sup(\bar{\gamma})}$.

[5] See [Dra74, Chapter 3, §5].

But the two are Δ_0-formulas on the parameters $\vec{Z}\restriction\alpha', \vec{A}\restriction\alpha', S\cap\alpha', M_{\bar\gamma}$ and $M_{\sup(\bar\gamma)}$, which are all elements of M_{β_0}. Therefore,

$$\langle M_{\beta_n}, \in, \vec{M}\restriction\beta_n\rangle \models \bar\gamma \text{ is the least ordinal such that } \{\vec{Z}\restriction\alpha', \vec{A}\restriction\alpha', S\cap\alpha'\}\subseteq M_\gamma,$$

so that $j_n^{-1}(\gamma) = \bar\gamma = j_0^{-1}(\gamma)$. □

Denote $\bar\gamma := j_0^{-1}(\gamma)$. Hence if we interpret $\dot\kappa, \dot\gamma, \dot{\vec{Z}}, \dot{S}, \dot{A}_k$ for $k\in\omega$ as $\alpha', \bar\gamma, \vec{Z}\restriction\alpha', S\cap\alpha', A_k\restriction\alpha'$ for $k\in\omega$, respectively, then $M_{\beta_{n+1}}$ believes that there are exactly n ordinals β such that $\langle M_\beta, \in, \vec{M}\restriction\beta\rangle\models T$ with $\dot{\vec{M}}$ interpreted as $\vec{M}\restriction\beta$, while M_{β_n} believes that there are exactly $n-1$ such ordinals.

Thus, as the sequence $\vec{M}$ is $\subseteq$-increasing, it follows that for all $k<n<\omega$, $\beta_k<\beta_n$ and $j_n(M_{\beta_k}) = M_{\delta_k}$.

Subclaim 2.15.4.2. *$\beta' := \sup_{n\in\omega}\beta_n$ is equal to $\sup(S_{\alpha'})$.*

Proof. For each $n<\omega$, as $\mathrm{clps}(\mathfrak{B}_{n,\alpha'}) = \langle M_{\beta_n}, \in, \ldots\rangle$, the proof of Subclaim 2.15.3.2, establishing that $\beta(\alpha)\in S_\alpha$, makes clear that $\beta_n\in S_{\alpha'}$.

We now turn to argue that $\beta'\notin S_{\alpha'}$ by showing that $\langle M_{\beta'}, \in\rangle \not\models \mathsf{ZF}^-$. Note that $\{\beta_n \mid n<\omega\}$ is a definable subset of β' since it can be defined as the first ω ordinals to satisfy Clauses (a)–(g), replacing κ by α'. So if $\langle M_{\beta'}, \in\rangle$ were to model ZF^-, we would get that $\sup_{n<\omega}\beta_n$ is in $M_{\beta'}$, contradicting the fact that $M_{\beta'}\cap\mathrm{OR} = \beta'$.

Next, suppose that $\beta>\beta'$ and $\beta\in S_{\alpha'}$. In particular, $\langle M_\beta, \in\rangle\models\mathsf{ZF}^-$, and $\langle\beta_n \mid n<\omega\rangle\in M_\beta$, so that $\langle M_{\beta_n}\mid n\in\omega\rangle\in M_\beta$. We will reach a contradiction to Clause (iii) of the definition of $S_{\alpha'}$, asserting, in particular, that $S\cap\alpha'$ is stationary in $\langle M_\beta, \in\rangle$.

For each $n<\omega$, we have that $\langle M_{\delta_{n+1}}, \in, \vec{M}\restriction\delta_{n+1}\rangle\models\Psi(C_n, \delta_n, \vec{\mathfrak{B}}_n, \kappa)$, where $\Psi(C_n, \delta_n, \vec{\mathfrak{B}}_{\mathfrak{n}}, \kappa)$ is the conjunction of the following two formulas:

- $C_n = \{\alpha<\kappa \mid B_{n,\alpha}\cap\kappa = \alpha\}$, and
- $\vec{\mathfrak{B}}_n$ is the $<_\Theta$-least witness for $\mathsf{LCC}(\kappa)$ at δ_n with respect to $\mathcal{F}_\emptyset$.

Therefore, for $\overline{C_n} := j_{n+1}^{-1}(C_n)$ and $\overline{\mathfrak{B}_n} := j_{n+1}^{-1}(\vec{\mathfrak{B}}_n)$, we have

$$\langle M_{\beta_{n+1}}, \in, \vec{M}\restriction\beta_{n+1}\rangle\models\Psi(\overline{C_n}, \beta_n, \overline{\mathfrak{B}_n}, \alpha').$$

In particular, $\overline{C_n} = j_{n+1}^{-1}(C_n) = C_n\cap\alpha'$. Recalling that $\alpha' = \min((\bigcap_{n\in\omega}C_n)\cap S)$, we infer that $\bigcap_{n<\omega}\overline{C_n}$ is disjoint from $S\cap\alpha'$. Thus, to establish that $S\cap\alpha'$ is nonstationary, it suffices to verify the two:

1. $\langle\overline{C_n}\mid n<\omega\rangle$ belongs to M_β;
2. for every $n<\omega$, $\langle M_\beta, \in\rangle\models\overline{C_n}$ is a club in α'.

As $\langle M_{\beta_n}\mid n\in\omega\rangle\in M_\beta$, we can define $\langle\overline{\mathfrak{B}}_n\mid n\in\omega\rangle$ using that, for all $n\in\omega$,

$$\langle M_{\beta_{n+1}}, \in, \vec{M}\restriction\beta_{n+1}\rangle\models \text{“}\overline{\mathfrak{B}}_n \text{ is the } <_\Theta\text{-least to witness } \mathsf{LCC}(\alpha') \text{ at } \beta_n$$
$$\text{with respect to } \langle(\emptyset,1),(a\cap\alpha',1),(S\cap\alpha',1),(\vec{Z}\restriction\alpha',2)\rangle\text{”}.$$

This takes care of Clause (1), and shows that $\langle M_{\beta_{n+1}}, \in\rangle\models\overline{C_n}$ is a club in α'. Since M_β is transitive and the formula expressing that $\overline{C_n}$ is a club is Δ_0, we have also taken care of Clause (2). □

It follows that $\alpha'\in D$ and $f(\alpha') = \sup(S_{\alpha'}) = \beta'$.[6] Finally, as, for every $n<\omega$, we have

$$\langle\alpha', \in, \vec{A}\restriction\alpha'\rangle\models_{M_{\beta_n}}\forall X\exists Y\varphi,$$

we infer that $N_{\alpha'} = M_{f(\alpha')} = M_{\beta'} = \bigcup_{n\in\omega}M_{\beta_n}$ is such that

$$\langle\alpha', \in, \vec{A}\restriction\alpha'\rangle\models_{N_{\alpha'}}\forall X\exists Y\varphi.$$ □

[6]Notice that the argument of this claim also showed that D is stationary.

This completes the proof of Theorem 2.15. □

As a corollary we have found a strong combinatorial axiom that holds everywhere (including at ineffable sets) in canonical models of Set Theory (including Gödel's constructible universe).

Corollary 2.16. *If $L[E]$ is an iterable extender model with Jensen λ-indexing having no subcompact cardinals, then for every regular uncountable cardinal κ and every stationary $S \subseteq \kappa$, $\mathrm{Dl}^*_S(\Pi^1_2)$ holds.*

Proof. By Lemma 2.13 and Theorem 2.15. □

3. Universality of inclusion modulo nonstationary

Throughout this section, κ denotes a regular uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$. Here, we will be proving Theorems B and C. Before we can do that, we shall need to establish a *transversal lemma*, as well as fix some notation and coding that will be useful when working with structures of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$.

Proposition 3.1 (Transversal lemma)**.** *Suppose that $\langle N_\alpha \mid \alpha \in S\rangle$ is a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence, for a given stationary $S \subseteq \kappa$. For every Π^1_2-sentence ϕ, there exists a transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$ satisfying the following.*

For every $\eta \in \kappa^\kappa$, whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, there are stationarily many $\alpha \in S$ such that

(i) *$\eta_\alpha = \eta \restriction \alpha$, and*
(ii) *$\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.*

Proof. Let $c : \kappa \times \kappa \leftrightarrow \kappa$ be some primitive-recursive pairing function. For each $\alpha \in S$, fix a surjection $f_\alpha : \kappa \to N_\alpha$ such that $f_\alpha[\alpha] = N_\alpha$ whenever $|N_\alpha| = |\alpha|$. Then, for all $i < \kappa$, as $f_\alpha(i) \in N_\alpha$, we may define a set η^i_α in N_α by letting

$$\eta^i_\alpha := \begin{cases} \{(\beta,\gamma) \in \alpha \times \alpha \mid c(i, c(\beta,\gamma)) \in f_\alpha(i)\} & \text{if } i < \alpha;\\ \emptyset & \text{otherwise.}\end{cases}$$

We claim that for every Π^1_2-sentence ϕ, there exists $i(\phi) < \kappa$ for which $\langle \eta^{i(\phi)}_\alpha \mid \alpha \in S\rangle$ satisfies the conclusion of our proposition. Before we prove this, let us make a few reductions.

First of all, it is clear that for every Π^1_2-sentence $\phi = \forall X \exists Y \varphi$, there exists a large enough $n' < \omega$ such that all predicates mentioned in φ are in $\{\epsilon, \mathbb{X}, \mathbb{Y}, \mathbb{A}_n \mid n < n'\}$. So the only structures of interest for ϕ are in fact $\langle \alpha, \in, (A_n)_{n<n'}\rangle$, where $\alpha \le \kappa$. Let $m' := \max\{m(\mathbb{A}_n) \mid n < n'\}$. Then, by a trivial manipulation of φ, we may assume that the only structures of interest for ϕ are in fact $\langle \alpha, \in, A_0\rangle$, where $n' \le \alpha \le \kappa$ and $m(\mathbb{A}_0) = m' + 1$.

Having the above reductions in hand, we now fix a Π^1_2-sentence $\phi = \forall X \exists Y \varphi$ and positive integers m and k such that the only predicates mentioned in φ are in $\{\epsilon, \mathbb{X}, \mathbb{Y}, \mathbb{A}_0\}$, $m(\mathbb{A}_0) = m$ and $m(\mathbb{Y}) = k$.

Claim 3.1.1. *There exists $i < \kappa$ satisfying the following. For all $\eta \in \kappa^\kappa$ and $A \subseteq \kappa^m$, whenever $\langle \kappa, \in, A\rangle \models \phi$, there are stationarily many $\alpha \in S$ such that*

(i) *$\eta^i_\alpha = \eta \restriction \alpha$, and*
(ii) *$\langle \alpha, \in, A \cap (\alpha^m)\rangle \models_{N_\alpha} \phi$.*

Proof. Suppose not. Then, for every $i < \kappa$, we may fix $\eta_i \in \kappa^\kappa$, $A_i \subseteq \kappa^m$ and a club $C_i \subseteq \kappa$ such that $\langle \kappa, \in, A_i\rangle \models \phi$, but, for all $\alpha \in C_i \cap S$, one of the two fails:

(i) $\eta^i_\alpha = \eta_i \restriction \alpha$, or
(ii) $\langle \alpha, \in, A_i \cap (\alpha^m)\rangle \models_{N_\alpha} \phi$.

Let

- $Z := \{c(i, c(\beta,\gamma)) \mid i < \kappa, (\beta,\gamma) \in \eta_i\}$,
- $A := \{(i, \delta_1, \ldots, \delta_m) \mid i < \kappa, (\delta_1, \ldots, \delta_m) \in A_i\}$, and
- $C := \triangle_{i<\kappa}\{\alpha \in C_i \mid \eta_i[\alpha] \subseteq \alpha\}$.

Fix a variable i that does not occur in φ. Define a first-order sentence ψ mentioning only the predicates in $\{\epsilon, \mathbb{X}, \mathbb{Y}, \mathbb{A}_1\}$ with $m(\mathbb{A}_1) = 1 + m$ and $m(\mathbb{Y}) = 1 + k$ by replacing all occurrences of the form $\mathbb{A}_0(x_1, \ldots, x_m)$ and $\mathbb{Y}(y_1, \ldots, y_k)$ in φ by $\mathbb{A}_1(i, x_1, \ldots, x_m)$ and $\mathbb{Y}(i, y_1, \ldots, y_k)$, respectively. Then, let $\varphi' := \forall i(\psi)$, and finally let $\phi' := \forall X \exists Y \varphi'$, so that ϕ' is a Π^1_2-sentence.

A moment reflection makes it clear that $\langle \kappa, \in, A\rangle \models \phi'$. Thus, let S' denote the set of all $\alpha \in S$ such that all of the following hold:

(1) $\alpha \in C$;
(2) $c[\alpha \times \alpha] = \alpha$;
(3) $Z \cap \alpha \in N_\alpha$;
(4) $|N_\alpha| = |\alpha|$.
(5) $\langle \alpha, \in, A \cap (\alpha^{m+1})\rangle \models_{N_\alpha} \phi'$;

By hypothesis, S' is stationary. For all $\alpha \in S'$, by Clauses (3) and (4), we have $Z \cap \alpha \in N_\alpha = f_\alpha[\alpha]$, so, by Fodor's lemma, there exists some $i < \kappa$ and a stationary $S'' \subseteq S' \setminus (i + 1)$ such that, for all $\alpha \in S''$:

(3') $Z \cap \alpha = f_\alpha(i)$.

Let $\alpha \in S''$. By Clause (5), we in particular have

(5') $\langle \alpha, \in, A_i \cap (\alpha^m)\rangle \models_{N_\alpha} \phi$.

Also, by Clause (1), we have $\alpha \in C_i$, and so we must conclude that $\eta_i \restriction \alpha \neq \eta^i_\alpha$. However, $\eta_i[\alpha] \subseteq \alpha$, and $Z \cap \alpha = f_\alpha(i)$, so that, by Clause (2),

$$\eta_i \restriction \alpha = \eta_i \cap (\alpha \times \alpha) = \{(\beta,\gamma) \in \alpha \times \alpha \mid c(i, c(\beta,\gamma)) \in f_\alpha(i)\} = \eta^i_\alpha.$$

This is a contradiction. □

This completes the proof of Proposition 3.1. □

Proposition 3.2. *Let α be an ordinal, and let X be a subset of $\alpha \times \alpha$. There is a first-order sentence ψ_{fnc} using X as a predicate such that:*

$$X \in \alpha^\alpha \text{ iff } \langle \alpha, \in, X\rangle \models \psi_{\text{fnc}}.$$

Proof. Let $\psi_{\text{fnc}} := \forall\beta\exists\gamma(X(\beta,\gamma) \wedge (\forall\delta(X(\beta,\delta) \to \delta = \gamma)))$. □

Proposition 3.3. *Let α be an ordinal. Suppose that ϕ is a Σ^1_1-sentence involving a predicate A and two binary predicates X_0, X_1. Denote $R_\phi := \{(X_0, X_1) \mid \langle \alpha, \in, A, X_0, X_1\rangle \models \phi\}$. Then there are Π^1_2-sentences $\psi_{\text{Reflexive}}$ and $\psi_{\text{Transitive}}$ such that:*

(1) $R_\phi \supseteq \{(\eta, \eta) \mid \eta \in \alpha^\alpha\}$ iff $\langle \alpha, \in, A\rangle \models \psi_{\text{Reflexive}}$;
(2) R_ϕ is transitive iff $\langle \alpha, \in, A\rangle \models \psi_{\text{Transitive}}$.

Proof. (1) Fix a first-order sentence ψ_{fnc} such that $X_0 \in \alpha^\alpha$ iff $\langle \alpha, \in, X_0\rangle \models \psi_{\text{fnc}}$. Now, let $\psi_{\text{Reflexive}}$ be $\forall X_0 \forall X_1((\psi_{\text{fnc}} \wedge (X_1 = X_0)) \to \phi)$.

(2) Fix a Σ^1_1-sentence ϕ' involving A and binary predicates X_1, X_2 and a Σ^1_1-sentence ϕ'' involving A and binary predicates X_0, X_2 such that

$$\{(X_1, X_2) \mid \langle \alpha, \in, A, X_1, X_2\rangle \models \phi'\} = R_\phi = \{(X_0, X_2) \mid \langle \alpha, \in, A, X_0, X_2\rangle \models \phi''\}.$$

Now, let $\psi_{\text{Transitive}} := \forall X_0 \forall X_1 \forall X_2((\phi \wedge \phi') \to \phi'')$. □

Definition 3.4. Denote by $\text{Lev}_3(\kappa)$ the set of level sequences in $\kappa^{<\kappa}$ of length 3:

$$\text{Lev}_3(\kappa) := \bigcup_{\tau<\kappa} \kappa^\tau \times \kappa^\tau \times \kappa^\tau.$$

Fix an injective enumeration $\{\ell_\delta \mid \delta < \kappa\}$ of $\mathrm{Lev}_3(\kappa)$. For each $\delta < \kappa$, we denote $\ell_\delta = (\ell^0_\delta, \ell^1_\delta, \ell^2_\delta)$. We then encode each $T \subseteq \mathrm{Lev}_3(\kappa)$ as a subset of κ^5 via:

$$T_\ell := \{(\delta, \beta, \ell^0_\delta(\beta), \ell^1_\delta(\beta), \ell^2_\delta(\beta)) \mid \delta < \kappa, \ell_\delta \in T, \beta \in \mathrm{dom}(\ell^0_\delta)\}.$$

We now prove Theorem C.

Theorem 3.5. *Suppose $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for a given stationary $S \subseteq \kappa$.*

For every analytic quasi-order Q over κ^κ, there is a 1-Lipschitz map $f : \kappa^\kappa \to 2^\kappa$ reducing Q to $\subseteq^S$.

Proof. Let Q be an analytic quasi-order over κ^κ. Fix a tree T on $\kappa^{<\kappa} \times \kappa^{<\kappa} \times \kappa^{<\kappa}$ such that $Q = \mathrm{pr}([T])$, that is,

$$(\eta, \xi) \in Q \iff \exists \zeta \in \kappa^\kappa\ \forall \tau < \kappa\ (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T.$$

By Proposition 3.2, for each $i < 3$, we may fix a first-order sentence ψ^i_{fnc} using binary predicates X_0, X_1, X_2, and a predicate A of arity 5, such that, for each $i < 3$,

$$X_i \in \kappa^\kappa \text{ iff } \langle \kappa, \in, A, X_0, X_1, X_2\rangle \models \psi^i_{\mathrm{fnc}}.$$

Now, define a first-order sentence φ_Q in the above-mentioned language to be the conjunction of four formulas: ψ^0_{fnc}, ψ^1_{fnc}, ψ^2_{fnc}, and

$$\forall\tau\exists\delta\forall\beta \in \tau[\exists\gamma_0\exists\gamma_1\exists\gamma_2(X_0(\beta,\gamma_0)\ \wedge\ X_1(\beta,\gamma_1)\ \wedge\ X_2(\beta,\gamma_2)\ \wedge\ A(\delta,\beta,\gamma_0,\gamma_1,\gamma_2))].$$

Let $A := T_\ell$. Evidently, for all $\eta, \xi, \zeta \in \mathcal{P}(\kappa \times \kappa)$, we get that

$$\langle \kappa, \in, A, \eta, \xi, \zeta\rangle \models \varphi_Q$$

iff ($\eta, \xi, \zeta \in \kappa^\kappa$) and (for all $\tau < \kappa$, there is $\delta < \kappa$ such that $\ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)$ is in T). Let $\phi_Q := \exists X_2(\varphi_Q)$. Then ϕ_Q is a Σ^1_1-sentence involving predicats A, X_0, X_1 for which the induced binary relation

$$R_{\phi_Q} := \{(\eta, \xi) \in (\mathcal{P}(\kappa \times \kappa))^2 \mid \langle \kappa, \in, A, \eta, \xi\rangle \models \phi_Q\}$$

coincides with the quasi-order Q. Now, appeal to Proposition 3.3 with ϕ_Q and A to receive the corresponding Π^1_2-sentences $\psi_{\mathrm{Reflexive}}$ and $\psi_{\mathrm{Transitive}}$. Then, consider the following two Π^1_2-sentences:

- $\psi^0_Q := \psi_{\mathrm{Reflexive}}\ \wedge\ \psi_{\mathrm{Transitive}}\ \wedge\ \phi_Q$, and
- $\psi^1_Q := \psi_{\mathrm{Reflexive}}\ \wedge\ \psi_{\mathrm{Transitive}}\ \wedge\ \neg(\phi_Q)$.

Let $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ be a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence. Appeal to Proposition 3.1 with the Π^1_2-sentence ψ^1_Q, to obtain a corresponding transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$. Note that we may assume that, for all $\alpha \in S$, $\eta_\alpha \in {}^\alpha\alpha$, as this does not harm the key feature of the chosen transversal.[7]

For each $\eta \in \kappa^\kappa$, let

$$Z_\eta := \{\alpha \in S \mid A \cap \alpha^5 \text{ and } \eta \restriction \alpha \text{ are in } N_\alpha\}.$$

Claim 3.5.1. *Suppose $\eta \in \kappa^\kappa$. Then $S \setminus Z_\eta$ is nonstationary.*

Proof. Fix primitive-recursive bijections $c : \kappa^2 \leftrightarrow \kappa$ and $d : \kappa^5 \leftrightarrow \kappa$. Given $\eta \in \kappa^\kappa$, consider the club D_0 of all $\alpha < \kappa$ such that:

- $\eta[\alpha] \subseteq \alpha$;
- $c[\alpha \times \alpha] = \alpha$;
- $d[\alpha \times \alpha \times \alpha \times \alpha \times \alpha] = \alpha$.

Now, as $c[\eta]$ is a subset of κ, by the choice $\vec{N}$, we may find a club $D_1 \subseteq \kappa$ such that, for all $\alpha \in D_1 \cap S$, $c[\eta] \cap \alpha \in N_\alpha$. Likewise, we may find a club $D_2 \subseteq \kappa$ such that, for all $\alpha \in D_2 \cap S$, $d[A] \cap \alpha \in N_\alpha$.

For all $\alpha \in S \cap D_0 \cap D_1 \cap D_2$, we have

[7]For any α such that η_α is not a function from α to α, simply replace η_α by the constant function from α to $\{0\}$.

- $c[\eta \restriction \alpha] = c[\eta \cap (\alpha \times \alpha)] = c[\eta] \cap c[\alpha \times \alpha] = c[\eta] \cap \alpha \in N_\alpha$, and
- $d[A \cap \alpha^5] = d[A] \cap d[\alpha^5] = d[A] \cap \alpha \in N_\alpha$.

As N_α is p.r.-closed, it then follows that $\eta \restriction \alpha$ and $A \cap \alpha^5$ are in N_α. Thus, we have shown that $S \setminus Z_\eta$ is disjoint from the club $D_0 \cap D_1 \cap D_2$. □

For all $\eta \in \kappa^\kappa$ and $\alpha \in Z_\eta$, let:

$$\mathcal{P}_{\eta,\alpha} := \{p \in \alpha^\alpha \cap N_\alpha \mid \langle \alpha, \in, A \cap \alpha^5, p, \eta \restriction \alpha \rangle \models_{N_\alpha} \psi_Q^0\}.$$

Finally, define a function $f : \kappa^\kappa \to 2^\kappa$ by letting, for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$,

$$f(\eta)(\alpha) := \begin{cases} 1 & \text{if } \alpha \in Z_\eta \text{ and } \eta_\alpha \in \mathcal{P}_{\eta,\alpha}; \\ 0 & \text{otherwise.} \end{cases}$$

Claim 3.5.2. *f is 1-Lipschitz.*

Proof. Let η, ξ be two distinct elements of κ^κ. Let $\alpha \le \Delta(\eta, \xi)$ be arbitrary.

As $\eta \restriction \alpha = \xi \restriction \alpha$, we have $\alpha \in Z_\eta$ iff $\alpha \in Z_\xi$. In addition, as $\eta \restriction \alpha = \xi \restriction \alpha$, $\mathcal{P}_{\eta,\alpha} = \mathcal{P}_{\xi,\alpha}$ whenever $\alpha \in Z_\eta$. Thus, altogether, $f(\eta)(\alpha) = 1$ iff $f(\xi)(\alpha) = 1$. □

Claim 3.5.3. *Suppose $(\eta, \xi) \in Q$. Then $f(\eta) \subseteq^S f(\xi)$.*

Proof. As $(\eta, \xi) \in Q$, let us fix $\zeta \in \kappa^\kappa$ such that, for all $\tau < \kappa$, $(\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T$. Define a function $g : \kappa \to \kappa$ by letting, for all $\tau < \kappa$,

$$g(\tau) := \min\{\delta < \kappa \mid \ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)\}.$$

As $(S \setminus Z_\eta)$, $(S \setminus Z_\xi)$ and $(S \setminus Z_\zeta)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi \cap Z_\zeta$. Consider the club $D := \{\alpha \in C \mid g[\alpha] \subseteq \alpha\}$. We shall show that, for every $\alpha \in D \cap S$, if $f(\eta)(\alpha) = 1$ then $f(\xi)(\alpha) = 1$.

Fix an arbitrary $\alpha \in D \cap S$ satisfying $f(\eta)(\alpha) = 1$. In effect, the following three conditions are satisfied:

(1) $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Reflexive}}$,
(2) $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Transitive}}$, and
(3) $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \eta \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

In addition, since α is a closure point of g, by definition of φ_Q, we have

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha \rangle \models \varphi_Q.$$

As $\alpha \in S$ and φ_Q is first-order,[8]

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha \rangle \models_{N_\alpha} \varphi_Q,$$

so that, by definition of ϕ_Q,

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q.$$

By combining the preceding with clauses (2) and (3) above, we infer that the following holds, as well:

(4) $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

Altogether, $f(\xi)(\alpha) = 1$, as sought. □

Claim 3.5.4. *Suppose $(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \setminus Q$. Then $f(\eta) \nsubseteq^S f(\xi)$.*

Proof. As $(S \setminus Z_\eta)$ and $(S \setminus Z_\xi)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi$. As Q is a quasi-order and $(\eta, \xi) \notin Q$, we have:

(1) $\langle \kappa, \in, A \rangle \models \psi_{\text{Reflexive}}$,
(2) $\langle \kappa, \in, A \rangle \models \psi_{\text{Transitive}}$, and
(3) $\langle \kappa, \in, A, \eta, \xi \rangle \models \neg(\phi_Q)$.

[8] N_α is transitive and rud-closed (in fact, p.r.-closed), so that $N_\alpha \models \mathsf{GJ}$ (see [Mat06, §Other remarks on GJ]). Now, by [Mat06, §The cure in GJ, proposition 10.31], **Sat** is Δ_1^{GJ}.

so that, altogether,

$$\langle\kappa,\in,A,\eta,\xi\rangle\models\psi^1_Q.$$

Then, by the choice of the transversal $\langle\eta_\alpha\mid\alpha\in S\rangle$, there is a stationary subset $S'\subseteq S\cap C$ such that, for all $\alpha\in S'$:

(1') $\langle\alpha,\in,A\cap\alpha^5\rangle\models_{N_\alpha}\psi_{\text{Reflexive}}$,
(2') $\langle\alpha,\in,A\cap\alpha^5\rangle\models_{N_\alpha}\psi_{\text{Transitive}}$,
(3') $\langle\alpha,\in,A\cap\alpha^5,\eta\restriction\alpha,\xi\restriction\alpha\rangle\models_{N_\alpha}\neg(\phi_Q)$, and
(4') $\eta_\alpha=\eta\restriction\alpha$.

By Clauses (3') and (4'), we have that $\eta_\alpha\notin\mathcal{P}_{\xi,\alpha}$, so that $f(\xi)(\alpha)=0$.

By Clauses (1'),(2') and (4'), we have that $\eta_\alpha\in\mathcal{P}_{\eta,\alpha}$, so that $f(\eta)(\alpha)=1$.

Altogether, $\{\alpha\in S\mid f(\eta)(\alpha)>f(\xi)(\alpha)\}$ covers the stationary set S', so that $f(\eta)\not\subseteq^S f(\xi)$. □

This completes the proof of Theorem 3.5 □

Theorem B now follows as a corollary.

Corollary 3.6. *Assume that κ is a regular uncountable cardinal and* GCH *holds. Then there is a $({<}\kappa)$-directed-closed, κ^+–cc notion of forcing $\mathbb{P}$ such that, in $V^{\mathbb{P}}$,* GCH *holds and for every analytic quasi-order Q over κ^κ and every stationary $S\subseteq\kappa$, $Q\hookrightarrow_1\subseteq^S$.*

Proof. By Fact 2.12, Theorem 2.15 and Theorem 3.5. □

Remark 3.7. A quasi-order $\trianglelefteq$ over a space $X\in\{2^\kappa,\kappa^\kappa\}$ is said to be Σ^1_1-*complete* iff it is analytic and, for every analytic quasi-order Q over X, there exists a κ-Borel function $f:X\to X$ reducing Q to $\trianglelefteq$. As Lipschitz $\Longrightarrow$ continuous $\Longrightarrow$ κ-Borel, the conclusion of Corollary 3.6 gives that each $\subseteq^S$ is a Σ^1_1-complete quasi-order. Such a consistency was previously only known for S's of one of two specific forms, and the witnessing maps were not Lipschitz.

4. Concluding remarks

Remark 4.1. By [HKM18, Corollary 4.5], in L, for every successor cardinal κ and every theory (not necessarily complete) T over a countable relational language, the corresponding equivalence relation $\cong_T$ over 2^κ is either Δ^1_1 or Σ^1_1-complete. This dissatisfying dichotomy suggests that L is a singular universe, unsuitable for studying the correspondence between generalized descriptive set theory and model-theoretic complexities. However, using Theorem 3.5, it can be verified that the above dichotomy holds as soon as κ is a successor of an uncountable cardinal $\lambda=\lambda^{<\lambda}$ in which $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for both $S:=\kappa\cap\mathrm{cof}(\omega)$ and $S:=\kappa\cap\mathrm{cof}(\lambda)$. This means that the dichotomy is in fact not limited to L and can be forced to hold starting with any ground model.

Remark 4.2. Let $=^S$ denote the symmetric version of $\subseteq^S$. It is well known that, in the special case $S:=\kappa\cap\mathrm{cof}(\omega)$, $=^S$ is a κ-Borel* equivalence relation [MV93, §6]. It thus follows from Theorem 3.5 that if $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for $S:=\kappa\cap\mathrm{cof}(\omega)$, then the class of Σ^1_1 sets coincides with the class of κ-Borel* sets. Now, as the proof of [HK18, Theorem 3.1] establishes that the failure of the preceding is consistent with, e.g., $\kappa=\aleph_2=2^{2^{\aleph_0}}$, which in turn, by [Gre76, Lemma 2.1], implies that $\diamondsuit^*_S$ holds, we infer that the hypothesis $\mathrm{Dl}^*_S(\Pi^1_2)$ of Theorem 3.5 cannot be replaced by $\diamondsuit^*_S$. We thus feel that we have identified the correct combinatorial principle behind a line of results that were previously obtained under the heavy hypothesis of "$V=L$".

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